Empirical Likelihood Estimation and Testing in Covariance Structure Models

Stanislav Kolenikov* Yiyong Yuan†

February 17, 2009

Abstract

This paper applies empirical likelihood approach to estimation and testing in structural equation models. Relations to other estimation methods and optimality of the moment conditions as estimating equations are demonstrated. Model fit for over-identified models can be tested using the empirical likelihood ratio. The bootstrap can be used to estimate Bartlett correction factor to improve the accuracy of the asymptotic χ^2 distribution of this test. Estimation examples of real and simulated non-elliptically distributed data are provided.

AMS Classification: 62F12, 62H12, 62H15

Keywords: Bartlett correction, covariance structure models, empirical likelihood, simulation, structural equation models

*Corresponding author. Mailing address: 146 Middlebush Hall, Department of Statistics, University of Missouri, Colubmia, MO 65211-6100, USA. Email: kolenikovs@missouri.edu. Phone: +1 (573) 882-6376. Fax: +1 (573) 884-5524.

†324 Mumford Hall, Department of Agricultural Economics, University of Missouri, Columbia, MO 65211, USA.

1 Introduction

Structural equation modeling (SEM) is a set of statistical techniques to analyze linear relations among random variables including unobserved (latent) ones. They are widely used in social sciences, such as psychology, sociology, anthropology (MacCallum & Austin 2000), and gaining recognition in natural sciences such as ecology (Pugesek, Tomer & von Eye 2002). The first general formulations of SEM are due to the works of Jöreskog, Keesing and Wiley in the 1970s. Their framework is known as JKW or LISREL model and is widely used in current SEM work. There are also formulations proposed by Bentler & Weeks (1980) and McArdle & McDonald (1984) that are of similar representation capacity.

At its early stages, estimation and inference for SEM were made possible by contributions in econometrics (simultaneous equation models) and psychometrics (factor analysis). Lawley (1940), Anderson & Rubin (1956), Jöreskog (1969) proposed hypothesis testing in factor analysis. Bock & Bargmann (1966) introduced analysis of covariance. Jöreskog (1973) proposed normal theory maximum likelihood estimation of general SEMs. Jöreskog & Goldberger (1972) and Browne (1974, 1982, 1984) set up least squares estimation and its various modifications. Particularly, Browne proposed estimator under elliptical distribution and Arbitrary Distribution Function (ADF) estimator also known as the Weighted Least Squares (WLS) for distribution-free model. Bentler (1983) suggested using higher-order moments to identify model that is otherwise not identified by covariance based estimation approach. Muthén (1984) applied SEMs to categorical and censored data. Bollen (1996*a*) investigated a limited information, two stage least squares (2SLS) instrumental variables estimator for latent variable SEMs. For a survey of problems and approaches in structural equation modeling, see Bollen (1989).

All those estimation methods have their pros and cons. The maximum likelihood estimation is susceptible to biases, especially in the standard errors, when the distribution of the data deviates from multivariate normal. While analysis of the asymptotic robustness conditions (Satorra 1990, Satorra & Bentler 1990) and corrections to the standard errors (Satorra & Bentler 1994) are available, efficiency of the method is compromised. ADF estimation procedure appears appealing as it achieves asymptotic efficiency. However, Chou, Bentler & Satorra (1991), Muthén & Kaplan (1985, 1992), and Hu, Bentler & Kano (1992), among others, found that ADF performs poorly with small sample sizes or large model degrees of freedom. The search for the best estimation method continues. This paper proposes a new estimation method with strong backing in theoretical statistics and econometrics.

Empirical likelihood (EL) theory, as a non-parametric inference method, has been developed first for moments of i.i.d. data (Owen 1990), then for regression, i.e., non-i.i.d. data (Owen 1991), and then for arbitrary estimating equations (Qin & Lawless 1994), thus allowing one to incorporate auxiliary information in estimation procedures. Since then it has been extended by many researchers into a systematic theory (Owen 2001) and found a great amount of following in econometrics (Mittelhammer, Judge & Miller 2000, Imbens 2002, Kitamura 2007). Computational algorithms and corrections improving on numerical and small sample issues have been proposed (Hall & La Scala 1990, Owen 2001, Chen, Variyath & Abraham 2008).

We adopt empirical likelihood estimation and testing procedures for structural equation models. The general theory of EL for estimating equations allowed us to identify the optimal estimating equations which turned out to be simply the moment conditions. A test of model fit naturally arises from the EL theory, and the bootstrap can be used to approximate Bartlett correction factor. A simulation compared EL to maximum (quasi-)likelihood and least squares, and showed that the former indeed provides improved estimates when the normality conditions are violated.

The remainder of the paper is organized as follows. The theory of empirical likelihood is reviewed in Section 2, and the framework of structural equation modeling, in Section 3. Section 4 blends covariance structure models and empirical likelihood. Examples and simulations are given in Section 5. Some further extensions and related issues are overviewed in Section 6. Section 7 concludes.

2 Basics of empirical likelihood

The concept of empirical likelihood stems from MLE estimation of the distribution function. For i.i.d. data, a non-parametric likelihood can be constructed as a product of probabilities $\mathrm{Prob}(X = x) = F(x) - F(x-)$ of individual observations:

$$L(F) = \prod_{i=1}^{n} \big[F(x_i) - F(x_i-)\big]. \tag{1}$$

Maximization of this likelihood is maximizing the point probability mass of the observed sample. Let the empirical distribution function be

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[x_i \le x],$$

where $\mathbb{1}[\cdot]$ is a 0/1 indicator function, and inequality is understood component-wise for multivariate x. It is easy to check that F_n maximizes non-parametric likelihood (1) (Owen 2001, Theorem 2.1). Thus, F_n is the non-parametric MLE of F.

Analogous to the parametric likelihood ratio test, define empirical likelihood ratio

$$R(F) = \frac{L(F)}{L(F_n)}$$

and profile likelihood function

$$R(\theta) = \sup\{R(F) | T(F) = \theta, F \in \mathbb{F}\}, \tag{2}$$

where $T(F)$ is a functional of the distribution F. A broad range of distribution characteristics can be thought of functionals of F, including moments and quantiles. In this paper, we consider the second moments of the data in covariance structure models.

As in the fully parametric case, the likelihood ratio test based on $R(\theta)$ rejects for low values of $R(\theta)$, and confidence sets can be formed as

$$\{\theta | R(\theta) \ge r_0\},$$

where r_0 controls coverage.

As long as profile likelihood (2) plays the role of the regular likelihood for inferential purposes, it can also be used to generate point estimates. Define maximum empirical likelihood estimator (MELE) as

$$\tilde{\theta} = \arg\max_{\theta} R(\theta). \tag{3}$$

If the condition $T(F) = 0$ can be thought of or expressed as a set of estimating equations $\mathbb{E}_F\, m(X, \theta) = 0$, the estimation problem can be written as

$$\tilde{\theta} = \arg\max_{w,\theta} \{ \prod_{i=1}^{n} n w_i | \sum_{i=1}^{n} w_i m(x_i, \theta), w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \}. \tag{4}$$

A number of general results about the properties of the resulting estimates have been established by Qin & Lawless (1994).

Theorem 1 *Let $X_i \in \mathbb{R}^d$ be iid random vectors, and suppose that $\theta_0 \in \mathbb{R}^t$ is uniquely determined by $\mathbb{E}[m(X,\theta)] = 0$, where $m(X,\theta)$ takes values in $\mathbb{R}^{t+q}$, for $q \geq 0$. If there is a neighborhood Θ of θ_0 and a function $M(x)$ with $\mathbb{E}[M(X)] < \infty$ so that for all $\theta \in \Theta$:*

- $\mathbb{E}[\partial m(X,\theta_0)/\partial\theta]$ *has rank t,*
- $\mathbb{E}[m(X,\theta_0) m(X,\theta_0)']$ *is positive definite,*
- $\partial m(x,\theta)/\partial\theta$ *is continuous in* θ*,*
- $\partial^2 m(x,\theta)/\partial\theta\,\partial\theta'$ *is continuous in* θ*,*
- $\max\{\|m(x,\theta)\|^3, \|\partial m(x,\theta)/\partial\theta\|, \|\partial^2 m(x,\theta)/\partial\theta\,\partial\theta'\|\} \leq M(x)$*,*

then MELE is consistent and asymptotically normal with asymptotic variance $\mathbb{V}[\tilde{\theta}]$ satisfying

$$\lim_{n\to\infty} n\mathbb{V}(\tilde{\theta}) = \left\{ \mathbb{E}\,\big(\frac{\partial m}{\partial\theta}\big)' [\mathbb{E}(mm')]^{-1}\,\mathbb{E}\,\big(\frac{\partial m}{\partial\theta}\big) \right\}^{-1} \tag{5}$$

This asymptotic variance is at least as small as that of any estimator of the form $\mathbb{E}\,A(\theta)m(X,\theta) = 0$, where $A(\theta)$ is a $p \times (p+q)$ matrix with rank p for all values of θ. Furthermore,

$$-\,2\log(R(\theta_0)/R(\tilde{\theta})) \xrightarrow{d} \chi^2_t \tag{6}$$

and

$$-\,2\log(R(\tilde{\theta})) \xrightarrow{d} \chi^2_q \tag{7}$$

as $n \to \infty$.

While formally the problem (4) appears to be of maximization over $n+p-1$ parameters (as maximization is performed jointly over the $n-1$ weights and p parameters), the effective degrees of freedom turned out to be the same as in the classical likelihood problems (Wilks 1938). There are t degrees of freedom correspond to the model space and the number of parameters, and residual q degrees of freedom to test the goodness of fit of the model.

The regularity conditions are quite typical for M-estimation problems (Huber 1974, White 1982). They are comparable to the conditions of Wald (1948) if the likelihood scores $\partial l(\theta, X)/\partial\theta$ are taken as the estimating equations $m(X,\theta)$, and more restrictive than those of Huber (1967) in requiring existence and smoothness of higher order derivatives.

Finally, (5) is the smallest possible variance of an estimator obtained from the set of estimating equations $m(x, \theta)$, or semiparametric efficiency bound (Stein 1956, Godambe & Thompson 1984, Chamberlain 1987, Bickel, Klaassen, Ritov & Wellner 1998). Thus, MELE is asymptotically efficient. Newey & Smith (2004) have also established higher order properties of EL, and found it to generally have lower biases than other estimators in class of generalized method of moments (GMM) and generalized empirical likelihood (GEL) estimators. With appropriate bias corrections, it also has the smallest higher order variance.

Confidence intervals, and in general confidence sets, can be constructed by inversion of the empirical likelihood ratio results in Theorem 1. An asymptotic confidence set based on the χ^2 distribution is

$$\big\{\theta : R(\theta) > c_\alpha, \mathrm{Prob}[\chi^2_q < -2\ln c_\alpha] = 1 - \alpha\big\}. \tag{8}$$

The empirical likelihood confidence regions have the same rate of coverage accuracy as parametric likelihood, jackknife and the bootstrap methods. The coverage error decreases to 0 at the rate of $\frac{1}{n}$ as $n \to \infty$, and affords Bartlett correction (Barndorff-Nielsen & Cox 1994), reducing the error rate from

$$\mathrm{Prob}[-2\ln R(\theta_0) \le x] = \mathrm{Prob}[\chi^2_q \le x] + O(n^{-1})$$

to

$$\mathrm{Prob}[(1 + a/n)(-2\ln R(\theta_0)) \le x] = \mathrm{Prob}[\chi^2_q \le x] + O(n^{-2})$$

with Bartlett factor a. Bartlett-correctability was established by DiCiccio, Hall & Romano (1991) for exactly identified models, and by Chen & Cui (2007) for overidentifed models.

Similar higher order improvements are also possible with the bootstrap calibration (Hall & La Scala 1990, Hall & Horowitz 1996, Brown & Newey 2002), either by direct estimation of the distribution function of $-2\ln R(\theta)$, or by a bootstrap-based estimate of Bartlett factor a:

$$\tilde{R}(\tilde{\theta}_{MELE}) = \frac{q \ln R(\tilde{\theta}_{MELE})}{\frac{1}{B}\sum_{b=1}^{B} \ln R(\hat{\theta}^{(b)}_*)}, \tag{9}$$

where B is the number of the bootstrap replications, and $\hat{\theta}^{(b)}_*$ are the estimates obtained in b-th bootstrap replicate. Calibration of the Bartlett correction can be done with a few dozen bootstrap samples, while direct estimation of the tail probability might require hundreds and thousands of the bootstrap replications.

Some general computational details, including the recently proposed adjusted empirical likelihood (AEL) method (Chen et al. 2008), are given in Appendix B.

3 Covariance structure models

This section provides a brief description of covariance structure models, or structural equation models (SEM)[1], and the main estimation and inference procedures associated with them. They encompass factor analysis, regression, errors-in-variables, growth models, and simultaneous equations as special cases (Bollen 1989, Yuan & Bentler 2007).

[1]A more general version of SEM will also include models for the means of the variables. In this paper, we consider the mean structure to be saturated, and concentrate on the covariance structure.

A general structural equation model consists of a latent model relating the latent variables:

$$\boldsymbol{\eta} = \boldsymbol{\alpha}_\eta + \boldsymbol{B}\boldsymbol{\eta} + \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta}, \tag{10}$$

and a measurement model linking the latent and the observed variables:

$$\boldsymbol{y} = \boldsymbol{\alpha}_y + \boldsymbol{\Lambda}_y\boldsymbol{\eta} + \boldsymbol{\varepsilon}, \tag{11}$$

$$\boldsymbol{x} = \boldsymbol{\alpha}_x + \boldsymbol{\Lambda}_x\boldsymbol{\xi} + \boldsymbol{\delta}. \tag{12}$$

Here, ξ are exogenous latent variables, η are endogenous latent variables, x and y are indicators of the exogenous and endogenous latent variables, and error terms $\zeta, \varepsilon, \delta$ are uncorrelated with one another and ξ. The set of model parameters includes the intercepts $\boldsymbol{\alpha}$, the effect of exogenous variables on the endogenous ones $\boldsymbol{\Gamma}$, the effect of endogenous variables on one another $\boldsymbol{B}$, the loadings $\boldsymbol{\Lambda}_x, \boldsymbol{\Lambda}_y$, and variances and covariances of unrestricted elements of block diagonal $\mathbb{V}[(\boldsymbol{\xi}', \boldsymbol{\zeta}', \boldsymbol{\varepsilon}', \boldsymbol{\delta}')'] = \operatorname{diag}[\Phi, \Psi, \Theta_{\boldsymbol{\varepsilon}}, \Theta_{\boldsymbol{\delta}}]$. Equations (10) – (12) imply a certain covariance structure:

$$\Sigma(\theta) = \mathbb{V}\Big[\begin{pmatrix} x \\ y \end{pmatrix}\Big] = \mathbb{V}\Big[\begin{pmatrix} \boldsymbol{\Lambda}_x\Phi\boldsymbol{\Lambda}_x' + \Theta_{\boldsymbol{\delta}} & \boldsymbol{\Lambda}_x\Phi\boldsymbol{\Gamma}'M' \\ M\boldsymbol{\Gamma}\Phi\boldsymbol{\Lambda}_x' & M\boldsymbol{\Gamma}\Phi\boldsymbol{\Gamma}'M' + M\Psi M' + \Theta_{\boldsymbol{\varepsilon}} \end{pmatrix}\Big], \quad M = \boldsymbol{\Lambda}_y(I - \boldsymbol{B})^{-1}. \tag{13}$$

Appropriateness, or goodness of fit, of the researcher's model is assessed via various tests of the null hypothesis

$$H_0 : \operatorname{Cov}(z) = \Sigma(\theta) \text{ vs. } H_1 : \operatorname{Cov}(z) \neq \Sigma(\theta), \tag{14}$$

where $z = (x', y')$ is the vector of stacked observed variables. If the null is rejected, the interpretation is that the covariance of the data cannot be approximated by the class of covariance matrices $\Sigma(\theta)$.

As most nonlinear models, SEMs face identification issues. A model is said to be identified if no two sets of parameters can generate the same likelihood. In the context of covariance structure models, it is usually weakened to the requirement that no two sets of parameters can generate the same covariance matrix. Identification by higher moments, although technically possible (Bentler 1983), typically requires huge sample sizes. Hence, the parameter identification problem can be cast in terms of inverting the (components of) implicit function $\Sigma(\theta)$ given by (13). If there are just enough conditions to identify all the elements of θ, the model is said to be exactly identified. If there is at least one θ that has more than one condition to be determined from, the model is said to be over-identified, and goodness of fit becomes possible (and usually desirable). For a detailed discussion on identification, see Bollen (1989).

Among the estimation methods for structural equation models, maximum likelihood estimation and variations of the least squares procedures are the two dominant ones.

The maximum likelihood estimator (MLE), also referred to as full information maximum likelihood (FIML) estimator, is defined as

$$\arg\min_\theta F_{ML}(\theta, S) = \arg\min\{\log|\Sigma(\theta)| + \operatorname{tr}[S\Sigma^{-1}(\theta)] - \log|S| - \dim(x', y')'\}, \tag{15}$$

where S is the MLE or unbiased estimator of the variance-covariance matrix without any structure imposed (i.e., the sample covariance matrix). It can be obtained either by assuming the multivariate normality of observed variables or by the Wishart distribution of the unbiased estimator of variance-covariance matrix. The objective function in (15) is the likelihood ratio against a saturated unstructured model normalized per observation. MLE has a number of well known good properties. It is scale free, consistent and asymptotically efficient, provided that the model is correct in both structural and distributional specifications. However, when the distribution of the data is not multivariate normal, MLE loses efficiency, and the covariance matrix of the parameter estimates is no longer the inverse of the information matrix (Satorra 1990, Satorra & Bentler 1994).

The goodness of fit test is available as the likelihood ratio test of (14) against a saturated model that does not assume any structure. It can also be computed equivalently as $T = (n-1)F_{ML}(\hat{\theta}, S)$. Under asymptotic robustness conditions (Satorra 1990), including the simplest case of multivariate normality, it has the asymptotic χ^2_q distribution where the degrees of freedom of the model $q = p^* - t$ is the difference between the total number of moments $p^* = \dim(z)(\dim(z)+1)/2$ and the number of parameters t. When the asymptotic robustness conditions are violated, its distribution becomes a mixture of χ^2_1 distributions with weights different from 1. Satterthwaite approximation to this distribution was proposed by Satorra & Bentler (1994) and is widely used in practice.

Alternatively, the distribution of T under the null can be approximated by the bootstrap. To make the data conform to the null hypothesis, Beran & Srivastava (1985) and Bollen & Stine (1992) proposed to augment them as

$$\tilde{X} = \Sigma(\hat{\theta})^{1/2} S^{-1/2} X, \tag{16}$$

where $\hat{\theta}$ is a valid value of the parameter vector usually taken to be an estimator obtained from any of the available methods. With this transformation, $\tilde{X}$ is guaranteed to have the correct model structure in the bootstrap population while retaining the multivariate kurtosis of the original data. Hence, the bootstrap distribution of the test statistic gives an approximation to the distribution of the test statistic under the null.

Different versions of the least squares procedures minimize the distance between the sample covariance matrix and the model implied covariance matrix. Unweighted Least Squares (ULS) estimator minimizes the square loss, or the Frobenius norm of the residual matrix:

$$\hat{\theta}_{ULS} = \arg\min_{\theta} \{ \frac{1}{2} \operatorname{tr}[(S - \Sigma(\theta))^2] \}.$$

Browne (1982) established consistency and provided inferential procedures.

The Generalized Least Squares (GLS) estimator is defined as

$$\hat{\theta}_{GLS} = \arg\min_{\theta} \frac{1}{2} \operatorname{tr}\{ [(\Sigma(\theta) S^{-1} - I)]^2 \}.$$

For correctly specified model structure, the GLS estimator is asymptotically equivalent to MLE (Lee & Jennrich 1979).

The most general formulation is the Weighted Least Squares (WLS) estimator, also titled Arbitrary Generalized Least Squares (AGLS) in Yuan & Bentler (2007), defined as

$$\hat{\theta}_{WLS} = \arg\min_{\theta} F_{WLS}(S, \theta, W) = \arg\min_{\theta} \{ [s - \sigma(\theta)]' W^{-1} [s - \sigma(\theta)] \}, \tag{17}$$

where $s = \operatorname{vech}(S)$ and $\sigma(\theta) = \operatorname{vech}[\Sigma(\theta)]$ are non-redundant vectorizations of the two covariance matrices (Magnus & Neudecker 1999), and W^{-1} is a positive definite (possibly random) weight matrix.

By a choice of the weight matrix W, the ULS or GLS methods can be reproduced, so WLS is the most general formulation of the minimal distance estimation methods. Indeed,

$$\begin{aligned}\operatorname{tr}\bigl\{[(S-\Sigma(\theta))V^{-1}]^2\bigr\} &= \operatorname{tr}\bigl[V^{-1}(S-\Sigma(\theta))V^{-1}(S-\Sigma(\theta))\bigr] \\ &= \operatorname{vec}'(S-\Sigma(\theta))(V^{-1}\otimes V^{-1})\operatorname{vec}(S-\Sigma(\theta)) \\ &= \operatorname{vech}'(S-\Sigma(\theta))D'(V^{-1}\otimes V^{-1})D\operatorname{vech}(S-\Sigma(\theta))\end{aligned}$$

by the properties of Kronecker products, and definition of the duplication matrix D (Magnus & Neudecker 1999). Here, $V = I$ for ULS and $V = S^{-1}$ for GLS. Thus, most of the modern treatments of SEM such as Yuan & Bentler (2007) are based on WLS.

As shown by Browne (1984), if W is chosen to be the asymptotic sample covariance matrix of s or a consistent estimator of it, the WLS is consistent and asymptotically efficient as long as the 4-th order moments of the observed variables are finite. If such W is used in estimation, it is known as the Asymptotically Distribution Free (ADF) estimation method. It is capable of achieving asymptotic efficiency for arbitrary distributions satisfying the regularity conditions of sufficiently many moments. However, as shown by Chou et al. (1991), Muthén & Kaplan (1985, 1992), and Hu et al. (1992), its performance under small sample size or large model degree of freedom is poor, and beneficial asymptotic properties are expressed only for sample sizes more than $10,000$.

Finally, a limited information estimation method that concentrates on one or few parameters at a time is two-stage least squares estimator with instrumental variables (Bollen 1996*a*, Bollen 1996*b*, Bollen 2001). This method allows to estimate regression-type coefficients, including latent variable loadings on indicators, and provides useful specification tests. Sargan (1958) test checks instrument validity, and Hausman (1978) test checks validity of the model as a whole by comparing 2SLS estiamtes with another estimator, usually the MLE.

The 2SLS method imposes identification condition of setting one of the latent variable loadings to 1. It can be assumed without loss of generality that it is the first indicator of a latent variable. Assume also for simplicity that all variables are considered in their deviations from the (sample) means, and all intercepts in (10)–(12) are zeroes. Consider the first exogenous and endogenous latent variables, so that $x_1, x_2, \ldots, x_k$ are indicators of ξ_1, and $y_1, y_2, \ldots, y_l$ are indicators of η_1. By using the above scaling convention, express

$$\xi_1 = x_1 - \delta_1, \quad \eta_1 = y_1 - \varepsilon_1.$$

Then the regression equations for other indicators can be written as

$$x_2 = \lambda_{x21}x_1 + \delta_2 - \lambda_{x21}\delta_1 \equiv \lambda_{x21}x_1 + \tilde{\delta}_2, \tag{18}$$

$$\ldots$$

$$x_k = \lambda_{xk1}x_1 + \delta_k - \lambda_{xk1}\delta_1 \equiv \lambda_{xk1}x_1 + \tilde{\delta}_k. \tag{19}$$

Estimation of say (18) by OLS will lead to biased results since the regressor x_1 is correlated with the composite error $\tilde{\delta}_2$ (they both contain δ_1). However, one can find instrumental variables to estimate (18) by a common econometric technique of two stage least squares (Davidson & MacKinnon 1993, Wooldridge 2002). The set of instrumental variables must satisfy the following conditions: (i) the instruments are correlated with the regressors; (ii) the instruments are uncorrelated with the (composite) error term; (iii) there should be at least as many instruments as there are regressors. If measurement errors $\boldsymbol{\delta}$ are uncorrelated, the remaining indicators of ξ_1, namely $x_3, \dots, x_k$ can be used as instruments in regression equation (18). Indeed, $\mathrm{Cov}(x_1, x_3) = \lambda_{x31}\phi_{11} \neq 0$, $\mathrm{Cov}(x_3, \tilde{\delta}_2) = \mathbb{E}[(\lambda_{x31}\xi_1 + \delta_3, \delta_2 - \lambda_{x21}\delta_1] = 0$, and similarly for other variables. Since the regression equation only contains one explanatory variable x_1, there must be at least one instrument to satisfy condition (iii). Similarly, other loadings $\lambda_{x31}, \dots, \lambda_{xk1}$ for indicators of latent variables can be estimated.

To estimate the coefficients of regression between two latent variables, assume variables ξ_1 and η_1 are related as

$$\eta_1 = \gamma_{11}\xi_1 + \zeta_1.$$

Then the corresponding equations in the scaling indicators are

$$y_1 = \gamma_{11}x_1 + \zeta_1 - \gamma_{11}\delta_1 \equiv \gamma_{11}x_1 + \tilde{\zeta}_1. \tag{20}$$

The instruments for this equation can again be chosen from the remaining indicators of ξ_1. However, indicators of η_1 cannot be used as instruments since they contain ζ_1.

4 Application of empirical likelihood theory to SEM

In this section, we apply the empirical likelihood concepts outlined in Section 2 to structural equation models of Section 3.

4.1 Optimal estimating equations

The estimating equations for maximum likelihood objective function (15) can be found using its differential (Magnus & Neudecker 1999) with respect to the implied moments matrix $\Sigma(\theta)$ and eventually the vector of parameters θ. See Appendix A for derivations. The resulting maximum likelihood estimating equations, or score equations, are given by

$$\dot{\sigma}(\theta)W_{NT}^{-1}(s - \sigma(\theta)) = 0 \tag{21}$$

where the normal theory weight matrix is

$$W_{NT}^{-1} = 2D'(\Sigma^{-1} \otimes \Sigma^{-1})D,$$

and $\dot{\sigma}(\theta)$ is the Jacobian of the transformation $\theta \mapsto \sigma(\theta) \equiv \mathrm{vech}\,\Sigma(\theta)$. Hence, the estimating equations are linear combinations of the generic moment conditions

$$m_{kl}(X, \theta) = s - \sigma(\theta), \quad 1 \le k \le l \le p, \tag{22}$$

where the coefficients for the linear combinations are provided by the normal theory weight matrix.

Using similar techniques, the estimating equations for the weighted least squares are given by

$$\dot{\sigma}(\theta)'W^{-1}(s - \sigma(\theta)) = 0,$$

with details in the Appendix. Hence, they also represent linear combinations of the moment conditions (22).

Let us now derive the estimating equations for the 2SLS method. Unlike the MLE or WLS estimating equations derived as first order conditions for certain optimization problems, 2SLS estimating equations can be obtained directly. For equation (18) and instruments $x_3, \ldots, x_k$, the estimating equations are

$$\sum_i [x_{i2} - \lambda_{x21} x_{i1}](x_{i3}, \ldots, x_{ik})' = 0, \tag{23}$$

with corresponding population analogues

$$\mathbb{E}[x_2 - \lambda_{x21} x_1](x_3, \ldots, x_k)' = 0 \tag{24}$$

or

$$\begin{aligned} \operatorname{Cov}(x_2, x_3) - \lambda_{x21} \operatorname{Cov}(x_1, x_3) &= 0 \\ &\vdots \\ \operatorname{Cov}(x_2, x_k) - \lambda_{x21} \operatorname{Cov}(x_1, x_k) &= 0 \end{aligned} \tag{25}$$

(recall that the intercepts were suppressed).

For equation (20) with instruments $x_3, \ldots, x_k$, the estimating equations are

$$\sum_i [y_{i1} - \gamma_{11} x_{i1}](x_{i3}, \ldots, x_{ik})' = 0, \tag{26}$$

with corresponding population analogues

$$\mathbb{E}[y_1 - \gamma_{11} x_1](x_3, \ldots, x_k)' = 0 \tag{27}$$

or

$$\begin{aligned} \operatorname{Cov}(y_1, x_3) - \gamma_{11} \operatorname{Cov}(x_1, x_3) &= 0 \\ &\vdots \\ \operatorname{Cov}(y_1, x_k) - \gamma_{11} \operatorname{Cov}(x_1, x_k) &= 0 \end{aligned} \tag{28}$$

Both systems (23) and (26) are obtained as linear combinations of the generic moment conditions (22). E.g., the first equation of (25) is obtained from the two moment conditions, $m_{13} = s_{13} - \lambda_3 \phi_{11}$ and $m_{23} = s_{23} - \lambda_2 \lambda_3 \phi_{11} = 0$, with weights $-\lambda_{x21}$ and 1, so that the implied moments disappear from the final expression $-\lambda_{x21} s_{13} + s_{23}$.

We now see that all the estimation methods outlined above are based on the moment conditions (22) with various linear combinations of those providing estimating equations for each particular method (and each particular parameter in 2SLS). Hence, the following result holds:

Corollary 1 *For the structural equations model (10)–(13), the empirical likelihood estimator based on moment conditions (22) is an asymptotically efficient estimator in the class of all estimators based on the second order moments only.*

In particular, asymptotic efficiency of MELE is at least as good as that of any WLS or 2SLS estimator.

The conditions of Theorem 1 are easy to interpret for SEM. The first condition requires that $\operatorname{rk} \dot\sigma(\theta_0) = p$, which is equivalent to local identifiability of the model. The next condition on positivity of $\mathbb{V}[s - \sigma(\theta)]$ will hold unless some moments are identical in case of perfect collinearity or redundancy of the variables. This also has to do with identification of the model. The derivatives for the smoothness conditions are given by Neudecker & Satorra (1991), and their continuity is easily verified as long as $|\Sigma(\theta)| > 0$. Finally, the conditions on the upper bounds of the estimating equations and their derivatives dictate that the sixth moment of the data is finite, which is a slightly more demanding condition than existence of the fourth order moments required for consistency and asymptotic efficiency of ADF.

Thus, the maximum empirical likelihood estimates for the structural equation model (10)–(12) are obtained as

$$\hat\theta_{MELE} = \arg\max_{\theta,w} \sum_{i=1}^{n} \ln(nw_i), \tag{29}$$

subject to constraints

$$\begin{aligned} &\sum_{i=1}^{n} w_i = 1, \\ &\sum_{i=1}^{n} w_i \big[(z_{ik} - \mu_k(\theta))(z_{il} - \mu_l(\theta)) - \sigma_{kl}(\theta)\big] = 0, \quad 1 \le k \le l \le p. \end{aligned} \tag{30}$$

If the means of the variables are not modeled, $\mu_i(\theta) = \bar x_i$. The dual saddle point formulation (see Appendix B) is

$$\hat\theta_{MELE} = \arg\max_{\theta} \min_{\lambda} \sum_{i=1}^{n} \ln \frac{1}{n} \frac{1}{1 + \lambda'\big[\operatorname{vech}(z_i - \mu(\theta))(z_i - \mu(\theta))' - \sigma(\theta)\big]}. \tag{31}$$

4.2 A test of model fit

In SEM analysis, substantive interest of the researcher usually lies in testing hypotheses (14) of whether her model is compatible with the data. In the EL framework, test of overall model can be constructed based on the profile likelihood

$$R(\theta) = \max\{\prod_{i=1}^{n} nw_i | \sum_{i=1}^{n} w_i \operatorname{vech}[Z_i Z_i' - \Sigma(\theta)] = 0, w_i \ge 0, \sum_{i=1}^{n} w_i = 1\},$$

By the result (7) of Theorem 1, it has a familiar χ^2_q distribution:

$$T_{MELE} = -2 \log R(\tilde\theta_{MELE}) \xrightarrow{d} \chi^2_q.$$

The degrees of freedom of the test is the number of overidentifying restrictions, and is the same as in the normal-theory based χ^2-difference test for multivariate normal data. However, EL-based test does not invoke any distributional assumptions, and does not need any corrections like those of Satorra & Bentler (1994) for the normal theory test statistic T.

An asymptotically equivalent test of model fit can be obtained via Lagrange multipliers λ. Imbens (2002) suggests the following form of the test statistic:

$$LM_{MELE} = n\hat{\lambda}'\hat{\Gamma}\hat{\lambda}, \tag{32}$$

where $\hat{\lambda}$ are the Lagrange multipliers at the optimum, and $\hat{\Gamma}$ is the estimator of the variance of estimating equations, e.g.,

$$\hat{\Gamma} = (s - \sigma(\hat{\theta}))(s - \sigma(\hat{\theta}))'.$$

This is the same matrix of empirical fourth moments that appears in ADF and Satorra & Bentler (1994) corrections.

5 Examples

In this section, we report the results of simulation undertaken to compare the performance of several estimation and testing methods, including EL and AEL. We also report the results of analysis based on a popular factor analysis data set.

5.1 Simulation study

For this simulation project, a confirmatory factor analysis model was considered:

$$x_k = \lambda_k \xi + \epsilon_k, \quad k = 1, \ldots, 6, \tag{33}$$

where all parameters λ_k, θ_k and ϕ were equal to one.

Three distributions were considered to study possible effects of skewness and kurtosis, either marginal or joint:

D1: $\xi \sim N(0,1)$, $\epsilon_k \sim N(0,1)$ for $k = 1,2,3$, ϵ_k is half-normal, scaled to have mean 0 and variance 1, with marginal skewness of $\sqrt{2}(4-\pi)/(\pi-2)^{3/2} = 0/407$ and kurtosis $8(\pi-3)/(\pi-2)^2 = 0.869$.

D2: $\xi \sim N(0,1)$, $\epsilon_k \sim N(0,1)$ for $k = 1,2,3$, ϵ_k is lognormal with mean of logs equal to $-0.5\ln(\mathrm{e}^1 - 1) - 0.5$ and variances of the logs equal to 1, scaled to have mean 0 and variance 1. The marginal skewness of the error terms is $(\mathrm{e}+2)\sqrt{(}\mathrm{e}-1) = 6.185$, and marginal kurtosis is $\mathrm{e}^4 + 2\mathrm{e}^3 + 3\mathrm{e}^2 - 6 = 110.9$.

D3: $\xi \sim N(0,1)$, $\epsilon_k \sim N(0,1)$ for $k = 1,2,3$, $(\epsilon_4, \epsilon_5, \epsilon_6)' \sim t_6$, multivariate t distribution, scaled to have unit marginal variances.

For distributions **D1** and **D2**, asymptotic robustness conditions (Satorra 1990) predict that the quasi-MLEs will be asymptotically efficient for all parameters except those describing the variances of the non-normal terms (i.e, θ_4 through θ_6). Distribution **D2** should be more problematic as it has higher skewness and kurtosis. Distribution **D3** violates those conditions, and the distribution-free nature of empirical likelihood methods would be expected to provide asymptotic efficiency gains.

For each distribution, sample sizes $n = 100, 200, 300, 500$ and 1000 typical for applications have been studied. For each sample size, at least 250 Monte Carlo replications were taken. For each Monte Carlo sample, the following estimation and testing procedures were performed, in this order:

1. ULS estimation, using the population values of the parameters as starting values;

2. quasi-MLE, or FIML, estimation, using a convex combination of ULS estimates (with weight 0.9) and population values (with weight 0.1) as starting values;

3. AEL estimation, using a convex combination of FIML estimates (with weight 0.8), ULS (with weight 0.1) and population values (with weight 0.1) as starting values;

4. EL estimation, using a convex combination of FIML and AEL estimates (with weights 0.4) and population parameters (with weight 0.2).

Huber (1967) sandwich variance estimator was used with all methods, with the necessary derivatives taken numerically.

The tables with simulation results are given in appendix C available externally[2]. Here, we provide a summary of results that we found to be interesting.

The degree of violation of multivariate normality is assessed in Table 2. The deviations from normality are notable enough to be detected by Mardia (1970) test even at the smallest sample sizes.

The quality of the asymptotic χ^2_9 approximation to the goodness of fit tests is explored in Table 3 using Kolmogorov-Smirnov test (Conover 1998, Kendall, Stuart, Ord & Arnold 1999). Quasi-likelihood ratio test, or FIML fit statistic, and likelihood ratio and score/LM tests for empirical likelihood and adjusted empirical likelihood are compared. Among the distributions satisfying asymptotic robustness conditions, the one with light tails (**D1**) produced valid χ^2 statistics for sample sizes of 500 and above, with the quasi-likelihood ratio test statistic T_{FIML} achieving asymptotic normality even faster. The heavy tailed distribution **D2** only yielded χ^2 approximation for the T_{FIML} statistic. All other test statistics did not perform well. However, additional simulation results (Section 5.1) suggest that the situation can be rectified using Bartlett-corrected tests.

We investigate three aspects of parameter estimation: bias, efficiency, and confidence interval coverage, each approximated by the corresponding Monte Carlo quantity. Each of the characteristics was analyzed according to the second order expansion:

$$\sqrt{n}(\hat{\theta}^r_{j,n} - \theta_j) = \beta_{b0} + \beta_{b1} n^{-1/2} + \epsilon_{bnr}, \tag{34}$$

$$n(\hat{\theta}^r_{j,n} - \bar{\hat{\theta}}_{j,n})^2 = \beta_{e0} + \beta_{e1} n^{-1/2} + \epsilon_{enr}, \tag{35}$$

$$\begin{aligned} \mathbb{I}\{\theta_j > \hat{\theta}^r_{j,n} - 1.645\text{s.e.}[\hat{\theta}^r_{j,n}]\} &= \beta_{l0} + \beta_{l1} n^{-1/2} + \epsilon_{lnr}, \\ \mathbb{I}\{\theta_j < \hat{\theta}^r_{j,n} + 1.645\text{s.e.}[\hat{\theta}^r_{j,n}])\} &= \beta_{r0} + \beta_{r1} n^{-1/2} + \epsilon_{rnr}. \end{aligned} \tag{36}$$

Here, $\hat{\theta}^r_{j,n}$ is the estimate of parameter θ_j in r-th Monte Carlo sample for the sample size n, and subindices b, e, l and r stand for the models for bias, efficiency and coverage on the left and right limits of the CI, respectively. The regression models for each characteristic was estimated simultaneously for all methods, and the standard errors were corrected for clustering with the Monte Carlo sample being the cluster. This representation allows for concise analytic results driven summary of the simulation (Skrondal 2000).

[2] Tables will be made available as a separate file on the authors' website(s).

Note that some parameters can be considered parallel, as some of the variables in the model can be permuted. Hence, the parallel sets are: loadings and variances of the variables with normal errors, λ_2–λ_3 and θ_2–θ_3; loadings and variances of the variables with skewed and/or heavy tailed errors, λ_4–λ_6 and θ_4–θ_6.

Table 4 summarizes the results for the bias of the estimators. There is little evidence of the higher order biases in estimation of the loadings, although this might be an issue of the power of the Monte Carlo study. Both EL and AEL seem to exhibit higher order (negative) biases in estimation of the unique variance parameters θ_k.

Table 5 summarizes the results for the variance of the estimators. While efficiency of ULS and quasi-MLE, surprisingly, is about the same on most occasions, the estimates of AEL and EL suggest higher efficiency. Individual differences do not appear to be significant at conventional levels, probably due to insufficient simulation study size, but overall F-tests reject the null that performance of the methods is the same within a given distribution. The distribution for which AEL and EL offer clear advantage is **D2**, with lognormal distribution of the error terms. In agreement with predictions of asymptotic robustness theory, FIML is not efficient for those parameters. Lower asymptotic variances of the empirical likelihood methods is somewhat offset by greater magnitude of higher order terms, which disagrees with predictions from Newey & Smith (2004).

Combining the biases and variances together into MSE, Table 6 shows that the leading term of the MSE of the EL/AEL estimates is generally lower than those of other methods, although the second order term is greater for EL/AEL than for other methods. The result is by and large driven by the variance part of MSE.

The confidence intervals coverage is reported in Table 7. One-sided CI nominal coverage is 95% (with the coverage error of $O(n^{-1/2})$), and two-sided CI nominal coverage is 90% (with the coverage error of $O(n^{-1})$). For all methods, the CI was constructed as the point estimate $\pm$ 1.645 times the reported standard error, and coverage in each tail was studied separately. Additionally, CIs based on profile EL (39) were studied for the distribution **D2**. The reported entries are deviations from the nominal coverage, so positive entries suggest overcoverage, and negative entries, undercoverage. Also, higher order terms show the effect of the sample size. For instance, a coefficient of -200 of the term $n^{-1/2}$ indicates that for a sample size of $n = 100$, the confidence interval has a coverage of only 75% vs. the target 95%. The lack of coverage only improves to 89% with the sample size of $n = 1000$. For two-sided coverage, the second term is of the order n^{-1}. The reported coefficient directly shows under- or overcoverage for the sample size of $n = 1000$, with coverage error being 10 times great for the sample size of $n = 100$.

Generally, confidence intervals for the variance parameters θ are the least accurate ones, especially for the variables were the distributions were tweaked (variables 4–6). That is not surprising in the light of asymptotic robustness theory.

For all distributions but **D1**, confidence intervals based on ULS estimation results are notably off the nominal coverage. Interestingly, in a lot of situations overcoverage occurs, so that the ULS CIs are too long. FIML did not have any major problems with the relatively simple distributions **D1** (that satisfies asymptotic robustness conditions and has relatively light tails) and **D3** (that violates asymptotic robustness, but produces a distribution that is relatively close to an elliptic one). EL and AEL methods performed similarly, and tended to have problems more often that FIML, although less often than ULS. Overcoverage was also observed with these methods. They also tended to have greater magnitudes of the second order terms. The main term and the second order term were of opposite signs in several cases, compensating one another in the moderate sample sizes. This may also be an indication of misspecified asymptotic expansion, e.g. when the term

$O(n^{-1/2})$ is absent in the expansion, while higher order terms such as $O(n^{-1})$ are present. Specifications with both $n^{-1/2}$ and n^{-1} as regressors were attempted, but the results were unstable because of very high multicollinearity between the two terms.

Contrary to the expectations, the profile empirical likelihood based CIs did not perform better in small to moderate size samples. While asymptotic coverage is on target, the small sample (second order) terms are generally greater in magnitude than with other methods, and indicate notable undercoverage. However, in the most difficult situation of the CI for the variance of the lognormal error terms in **D2**, the profile EL was the only method that had accurate coverage. All other methods overcovered on the left side and undercovered on the right side, with tendency to undercover for two-sided intervals.

A combination of the bootstrap based Bartlett correction with profiling the likelihood might improve the situation, but this approach was not studied in the current project.

Across most columns of Tables 4 through 7, F-tests are significant at conventional levels. It indicates that the estimators have notably different performance for most parameters and characteristics (34)–(36).

An additional simulation was used to gauge viability of the bootstrap to provide direct estimates of the goodness of fit statistic and/or Bartlett correction to it. Distribution **D3** that featured multivariate Student distribution of several error terms was used, with sample sizes of 100, 200, 500 and 1000. The number of Monte Carlo repetitions was 400, 600, 600 and 300, respectively. For each Monte Carlo sample, the following estimation and testing procedures were performed, in this order:

1. ULS estimation, using the population values as the starting point

2. (quasi-)MLE estimation, using a convex combination of ULS estimates (with weight 0.9) and population values (with weight 0.1)

3. EL estimation, using a convex combination of MLE (with weight 0.8), ULS (with weight 0.1) and population values (with weight 0.1)

4. $B = 200$ bootstrap replications of EL estimation from the augmented distribution rotated according to (16)

5. The direct bootstrap p-value was computed as

$$p_{BS} = \frac{1}{B} \sum_{b=1}^{B} \mathbb{1}[T_b^* > T_{EL}]$$

   where T_{EL} is the empirical likelihood ratio for the overall goodness of fit test achieved in the original Monte Carlo sample, and T_b^* is the EL ratio statistic obtained in b-th bootstrap sample

6. The Bartlett-corrected statistics were obtained according to (9) using either all 200 bootstrap replications, or just the first 50 bootstrap replications.

The simulation results are summarized in Table 8. The statistics T_{EL}, T_{FIML}, and the Bartlett-corrected versions of T_{EL} were compared to their asymptotic targets of χ^2_9, while the direct bootstrap estimate p_{BS} was compared to the uniform $U[0, 1]$ distribution. As expected, the likelihood ratio test for FIML rejects the model, since the distribution is misspecified. The empirical likelihood ratio only achieves its asymptotic χ^2 distribution at the largest sample sizes used. However, the bootstrap approach showed great performance, and Bartlett-corrected $\tilde{R}(\hat{\theta}_{MELE})$ had the desired pivotal distribution with as few as 50 bootstrap replications and sample size as small as $n = 100$.

5.2 Empirical illustration

We have used Holzinger-Swineford data analyzed by Jöreskog (1969) to compare several estimation methods[3]. The following variables were used in the analysis:

Factor 1	Factor 2	Factor 3
y_1, visual perception test	y_4, paragraph comprehension test	y_7, speeded addition test
y_2, spatial relations test	y_5, sentence completion test	y_8, speeded counting of dots in shape
y_3, lozenges	y_6, word meaning test	y_9, straight and curved caps

In order to avoid numeric optimization issues such as poorly condition matrices, the variables were scaled by 6, 4, 8, 3, 4, 7, 23, 20, 36, to make their empirical variances close to 1. The identification condition imposed on the system was to set the variances of the latent factors to 1, so that ϕ parameters represent factor correlations. Three methods were compared: maximum likelihood (FIML), ULS and empirical likelihood[4]. The results are reported in Table 1.

We would expect that the coefficient estimates would be comparable to one another, while the standard errors would be smaller for the empirical likelihood and larger for the less efficient ULS, with the (quasi) ML with Huber (1967) sandwich standard errors in between. Also, the model is known to fit quite poorly (Yuan & Bentler 2007), so the overall fit tests are expected to reject the model.

Those expectations were by and large confirmed by the analysis, although the relations between the estimated covariance matrices were not monotonic. For each of the differences $\hat{\mathbb{V}}[\hat{\theta}_{ML}] - \hat{\mathbb{V}}[\hat{\theta}_{EL}]$, $\hat{\mathbb{V}}[\hat{\theta}_{ULS}] - \hat{\mathbb{V}}[\hat{\theta}_{EL}]$, 15 out of 21 eigenvalues were positive, and 6 were negative and smaller in magnitude that the positive ones, showing a tendency for $\hat{\mathbb{V}}[\hat{\theta}_{EL}]$ matrix to be smaller than the other two. Some of the parameters that EL seemed to be fitting more poorly are those of the latent factor structure Φ.

The normal theory T statistic was 85.17, but due to severe non-normality, kurtosis correcting (Satorra & Bentler (1994) adjusted T) was computed and reported in the table. Both adjusted T and empirical likelihood ratio test safely rejected the model, as expected. The bootstrap from the null using rotated data (16) was used to calibrate the χ^2 test statistics for both the FIML method and the empirical likelihood, using the FIML and EL estimates for $\hat{\theta}$, respectively. $R = 500$ bootstrap replications were taken for each of the methods. The number of bootstrap samples where the empirical likelihood ratio for the correctly specified model exceeded 91.281 was 39, so the direct bootstrap p-value is 39/500 = 0.078 indicating a mild rejection. None of the bootstrap samples had resampled T statistic exceeding 85.17, hence, the the bootstrap p-value for the quasi-ML T is less than 0.002.

Bartlett correction factor can also be approximated by the bootstrap from the null distribution. The mean of the empirical likelihood ratios was 63.363, much larger than the degrees of freedom 24. Hence, the

[3] The data set is that available in R package MBESS.

[4] Empirical likelihood was implemented by direct optimization of weights in MATLAB, as well as through the dual problem in Stata. The results agreed to numeric accuracy.

Table 1: Factor analysis of Holzinger-Swineford data.

Parameter	FIML	ULS	EL
λ_{11}	0.899	0.976	0.800
	(0.101)	(0.106)	(0.133)
λ_{21}	0.498	0.489	0.443
	(0.0879)	(0.0804)	(0.0966)
λ_{31}	0.656	0.614	0.724
	(0.0807)	(0.0786)	(0.0911)
λ_{42}	0.990	1.006	1.040
	(0.0614)	(0.0622)	(0.0616)
λ_{52}	1.102	1.061	1.108
	(0.0548)	(0.0585)	(0.0519)
λ_{62}	0.917	0.941	0.936
	(0.0582)	(0.0639)	(0.0484)
λ_{73}	0.619	0.489	0.619
	(0.0857)	(0.0904)	(0.0954)
λ_{83}	0.731	0.631	0.697
	(0.0922)	(0.0832)	(0.0826)
λ_{93}	0.671	0.866	0.717
	(0.0985)	(0.0869)	(0.109)
ϕ_{12}	0.459	0.430	0.381
	(0.0734)	(0.0726)	(0.102)
ϕ_{23}	0.284	0.292	0.241
	(0.0853)	(0.0730)	(0.109)
ϕ_{13}	0.470	0.464	0.515
	(0.118)	(0.0736)	(0.159)

Parameter	FIML	ULS	EL
θ_1	0.549	0.405	0.676
	(0.157)	(0.179)	(0.182)
θ_2	1.134	1.143	1.176
	(0.112)	(0.116)	(0.111)
θ_3	0.844	0.898	0.746
	(0.101)	(0.0941)	(0.136)
θ_4	0.371	0.338	0.355
	(0.0504)	(0.0621)	(0.0495)
θ_5	0.446	0.534	0.403
	(0.0568)	(0.0709)	(0.0511)
θ_6	0.356	0.311	0.311
	(0.0466)	(0.0519)	(0.0401)
θ_7	0.797	0.941	0.761
	(0.0968)	(0.0873)	(0.0910)
θ_8	0.488	0.624	0.445
	(0.118)	(0.0941)	(0.0935)
θ_9	0.568	0.267	0.498
	(0.118)	(0.131)	(0.120)
χ^2 test	72.812		91.281
d.f.	21.3		24
p-value	$3 \cdot 10^{-8}$		$9 \cdot 10^{-10}$
Direct bootstrap percentile			
	< 0.002		0.078
Bootstrap-calibrated Bartlett corrected			34.574
p-value			0.075

Standard errors are in parentheses. First column: FIML criterion, Huber sandwich standard errors, adjusted Satorra-Bentler test. Second column: ULS criterion. Third column: empirical likelihood and empirical likelihood ratio test.

Bartlett-corrected empirical likelihood ratio is

$$\frac{q \ln R(\hat{\theta}_{MELE})}{\frac{1}{B}\sum_{b=1}^{B} \ln R(\hat{\theta}_*^{(b)})} = \frac{24 \cdot 91.281}{63.363} = 34.574,$$

where $\hat{\theta}_*^{(r)}$ is the MELE obtained in r-th bootstrap sample. It is reported in the last block of Table 1, and the corresponding p-value of 0.075 is similar to that of the bootstrap distribution tail probability.

The empirical likelihood weights w_i demonstrated an appreciable spread, with the minimal value of $0.619 \cdot 10^{-3}$, the maximum value of $22.23 \cdot 10^{-3}$ (c.f. $1/n = 3.32 \cdot 10^{-3}$), and coefficient of variation of 1.24. This quite notable variability of weights might be yet another indication of poor model fit. If the estimating equations were all correct, then the Lagrange multipliers λ are of order $O_p(n^{-1/2})$ (Qin & Lawless 1994), and hence the weights are $1/n(1 + O_p(n^{-1/2}))$. In this example, however, the variability of weights is quite pronounced. The EL method had to go a great length in approximating the empirical covariance matrix with the weighted implied one.

Estimation of the model with analytical derivatives in λ and numeric derivatives in θ took about 20 minutes on a Windows XP computer with 2 GHz processor.

6 Discussion

There is a number of further topics that can be studied with regards to empirical likelihood approach in structural equation modeling.

Estimators based on a set of estimating equations, like those in (22), have been studied extensively in econometrics. The method of combining several moment conditions together via a quadratic form is known as generalized method of moments (Hansen 1982, Hall 2005, GMM). The properties of GMM estimators, such as consistency, asymptotic normality and asymptotic efficiency, have been established in econometrics literature, and it has become the basic estimation framework within which many other estimation methods can be explained (Hayashi 2000). In SEM, the WLS is the analogue of GMM estimator, and ADF is the asymptotically efficient GMM estimator. As with ADF estimator, finite sample problems of GMM have also been noted in econometric literature (Hansen, Heaton & Yaron 1996). Relations between GMM and empirical likelihood have been studied extensively (Smith 1997, Imbens 2002, Bera & Bilias 2002, Newey & Smith 2004, Hall, Inoue, Jana & Shin 2007). Newey & Smith (2004) demonstrated that EL estimates have smaller biases than other estimation methods such as (different versions of) GMM. This accumulated body of literature will be helpful in formulating new estimators for SEM and investigating their properties.

There is a number of related information-theory based estimators, including Euclidean empirical likelihood (Owen 2001), exponential tilting (Kitamura & Stutzer 1997), generalized empirical likelihood (Smith 1997), and Cressie-Read divergence family (Cressie & Read 1984). All of those estimators can be entertained for SEM, as well.

The empirical likelihood framework can provide inference for derived quantities such as estimates of direct, indirect and total effects. Testing whether a direct effect is zero is usually available through a test on a single parameter. A test on whether an indirect or a total effect is zero can be obtained by forming the corresponding nonlinear combinations of parameters, setting this direct or indirect effect to zero, adding it as an additional estimating equation along with the main equations (22), and forming the empirical likelihood ratio against the base model. Alternatively, the confidence intervals for the indirect and total effects can be formed as mentioned above by finding the level points of the empirical likelihood ratio.

While we have covered the main estimation methods commonly used in structural equation modeling, other more exotic methods can also be entertained provided they result in closed form estimating equations. For instance, robust estimators and estimating equations implied by them (Moustaki & Victoria-Feser 2006, Siemsen & Bollen 2007) can be used instead of the usual second moments (22).

It is interesting to note that empirical likelihood naturally treats missing data in pairwise fashion. That is, if the value x_{ij} of j-th variable is missing in i-th observation, this will lead to exclusion of i-th observation only from the estimating equations referring to the missing variable. All pairs of non-missing variables will still be present in their respective estimating equations for that observation. The estimating approaches that use estimates of covariance matrices in their objective functions naturally tend to rely on listwise deletion, where observation will be lost if any of its entries are missing, unless special modifications of the standard estimation procedures are performed.

Other bootstrap procedures can be used for Bartlett correction calibration in place of the bootstrap from the rotated distribution (16). One possible bootstrap scheme is unequal probability bootstrap from the distribution with estimated empirical likelihood weights. Alternatively, the bootstrap samples can be taken from the original distribution, and empirical likelihood ratios can be formed against the empirical likelihood of the original sample.

Empirical likelihood procedures place considerable computing demands. Consider the data set with n observations, and the model with p observed variables and t parameters. Direct optimization of weights is a constrained maximization procedure with $n + t$ parameters, and optimization via the dual problem is a nested set of maximization over t parameters where each nested call is a minimization with respect to p^* parameters where $p^* = p(p + 1)/2$. Depending on the sample size and optimizers available to the end user, one or the other might be preferred. A generic implementation is available in GAUSS by Bruce Hansen[5]. From the authors' experience, pushing the problems beyond a dozen or so observed variables may require enormous computing power generally not available in desktop formats.

7 Conclusion

This paper proposed to use empirical likelihood framework for estimation and testing in structural equation and covariance structure models. Based on the asymptotic theory of empirical likelihood and its extensions, the method has been expected to produce asymptotically efficient estimates as well as to provide a test of fit that has a pivotal χ^2 distribution regardless of the underlying distribution. The only other method known to have this important property is Browne's (1984) ADF that usually needs very large sample sizes.

It has been demonstrated that the EL or AEL estimates based on the complete set of second order moments will be asymptotically efficient, and have asymptotic variance at least as small as those of MLE, WLS or 2SLS estimators. Based on higher order theory, we would expect EL/AEL estimators to have lower biases and better coverage of confidence intervals.

Simulation evidence confirmed some of those findings, but not all of them. When asymptotic robustness conditions are satisfied, estimation and testing based on quasi-MLE produced results that are arguably better

[5] http://www.ssc.wisc.edu/~bhansen/progs/progs_gmm.html

that those based on EL methods, with more accurate confidence intervals and asymptotic χ^2 distribution of the goodness of fit test providing reasonable approximation at lower sample sizes. However, when asymptotic robustness conditions are violated, only the EL/AEL methods produce χ^2 distribution, although the sample sizes might have to be in high hundreds of observations.

Limited simulation evidence also suggests that Bartlett corrections may bring the sample sizes required for asymptotic χ^2 distribution to be sufficiently accurate down to low hundreds. The number of bootstrap replications that can be used to calibrate the Bartlett factor may be as low as 50.

There was mixed evidence regarding performance of the empirical likelihood estimates. They tended to have higher order biases in our simulations, and while the reported variances and MSEs were lower than for other methods, they failed to reach "significance" with the current number of Monte Carlo replications.

The simulation results suggest that the best way to apply the EL procedure in SEM is to complement the main estimation with the bootstrap calibration of the Bartlett corrections, probably for both the overall goodness of fit test and the CIs for the parameters of primary interest. Then reasonably accurate results can be obtained for sample sizes as low as $n = 100$.

Acknowledgments

The authors would like to thank Kenneth Bollen, Douglas Miller, Victoria Savalei, Oksana Loginova, three referees and the associate editor for helpful comments. The errors and omissions are our responsibility. This work was supported by NSF grant SES-0617193 with funds from Social Security Administration.

References

Anderson, T. W. & Rubin, H. (1956), Statistical inference in factor analysis, *in* 'Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability', Vol. 5, pp. 111–150.

Azzalini, A. & Dalla Valle, A. (1996), 'The multivariate skew-normal distribution', *Biometrika* **83**(4), 715–726.

Barndorff-Nielsen, O. E. & Cox, D. R. (1994), *Inference and Asymptotics*, Monographs on Statistics and Applied Probability, Chapman & Hall/CRC, New York.

Bentler, P. M. (1983), 'Simultaneous equation systems as moment structure models', *Journal of Econometrics* **22**(1-2), 13–42.

Bentler, P. M. & Weeks, D. G. (1980), 'Linear structural equations with latent variables', *Psychometrika* **45**, 289–308.

Bera, A. K. & Bilias, Y. (2002), 'The MM, ME, ML, EL, EF and GMM approaches to estimation: a synthesis', *Journal of Econometrics* **107**(1-2), 51–86.

Beran, R. & Srivastava, M. S. (1985), 'Bootstrap tests and confidence regions for functions of a covariance matrix', *The Annals of Statistics* **13**(1), 95–115.

Bickel, P. J., Klaassen, C. A. J., Ritov, Y. & Wellner, J. A. (1998), *Efficient and Adaptive Estimation for Semiparametric Models*, Springer.

Bock, R. & Bargmann, R. (1966), 'Analysis of covariance structures', *Psychometrika* **31**(4), 507–534.

Bollen, K. A. (1989), *Structural Equations with Latent Variables*, Wiley, New York.

Bollen, K. A. (1996*a*), 'An alternative two stage least squares (2SLS) estimator for latent variable models', *Psychometrika* **61**(1), 109–121.

Bollen, K. A. (1996*b*), A limited-information estimator for LISREL models with and without heteroscedastic errors, *in* G. Marcoulides & R. Schumaker, eds, 'Advanced Structural Equation Modeling Techniques', Erlbaum, Mahwah, NJ, pp. 227–241.

Bollen, K. A. (2001), Two-stage least squares and latent variable models: Simultaneous estimation and robustness to misspecifications, *in* R. Cudeck, S. D. Toit & D. Sörbom, eds, 'Structural Equation Modeling: Present and Future, A Festschrift in Honor of Karl Jöreskog', Scientific Software, pp. 119–138.

Bollen, K. & Stine, R. (1992), 'Bootstrapping goodness of fit measures in structural equation models', *Sociological Methods and Research* **21**, 205–229.

Brown, B. W. & Newey, W. K. (2002), 'Generalized method of moments, efficient bootstrapping, and improved inference', *Journal of Business and Economic Statistics* **20**, 507–517.

Browne, M. W. (1974), ‘Generalized least squares estimators in the analysis of covariances structures’, *South African Statistical Journal* **8**, 1–24.

Browne, M. W. (1982), Covariance structures, *in* D. M. Hawkins, ed., ‘Topics in Multivariate Analysis’, Cambridge University Press, pp. 72–141.

Browne, M. W. (1984), ‘Asymptotically distribution-free methods for the analysis of the covariance structures’, *British Journal of Mathematical and Statistical Psychology* **37**, 62–83.

Chamberlain, G. (1987), ‘Asymptotic efficiency in estimation with conditional moment restrictions’, *Journal of Econometrics* **34**(3), 305–334.

Chen, J., Variyath, A. M. & Abraham, B. (2008), ‘Adjusted empirical likelihood and its properties’, *Journal of Computational and Graphical Statistics* **17**, 426–443.

Chen, S. X. & Cui, H. (2007), ‘On the second-order properties of empirical likelihood with moment restrictions’, *Journal of Econometrics* **141**(2), 492–516.

Chou, C., Bentler, P. & Satorra, A. (1991), ‘Scaled test statistics and robust standard errors for non-normal data in covariance structure analysis: a Monte Carlo study’, *British Journal of Mathematical and Statistial Psychology* **44**(2), 347–357.

Conover, W. (1998), *Practical nonparametric statistics*, Wiley Series in Probability and Statistics, John Wiley and Sons, New York.

Cressie, N. & Read, T. (1984), ‘Multinomial goodness-of-fit tests’, *Journal of the Royal Statistical Society Series B* **46**, 440–464.

Davidson, R. & MacKinnon, J. (1993), *Estimation and Inference in Econometrics*, Oxford University Press, New York.

DiCiccio, T., Hall, P. & Romano, J. (1991), ‘Empirical likelihood is Bartlett-correctable’, *The Annals of Statistics* **19**(2), 1053–1061.

Godambe, V. P. & Thompson, M. E. (1984), ‘Robust estimation through estimating equations’, *Biometrika* **71**(1), 115–125.

Hall, A. R. (2005), *Generalized Method of Moments*, Oxford University Press.

Hall, A. R., Inoue, A., Jana, K. & Shin, C. (2007), ‘Information in generalized method of moments estimation and entropy-based moment selection’, *Journal of Econometrics* **138**(2), 488–512.

Hall, P. & Horowitz, J. L. (1996), ‘Bootstrap critical values for tests based on generalized-method-of-moments estimators’, *Econometrica* **64**(4), 891–916.

Hall, P. & La Scala, B. (1990), ‘Methodology and algorithms of empirical likelihood’, *International Statistical Review / Revue Internationale de Statistique* **58**(2), 109–127.

Hansen, L. P. (1982), 'Large sample properties of generalized method of moments estimators', *Econometrica* **12**, 347–360.

Hansen, L. P., Heaton, J. & Yaron, A. (1996), 'Finite-sample properties of some alternative GMM estimators', *Journal of Business & Economic Statistics* **14**(3), 262–280.

Hausman, J. (1978), 'Specification tests in econmetrics', *Econometrica* **46**, 1251–1271.

Hayashi, F. (2000), *Econometrics*, Princeton University Press, Princeton, NJ.

Hu, L. T., Bentler, P. M. & Kano, Y. (1992), 'Can test statistics in covariance structure analysis be trusted?', *Psychological Bulletin* **112**(2), 351–362.

Huber, P. (1967), The behavior of the maximum likelihood estimates under nonstandard conditions, *in* 'Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability', Vol. 1, University of California Press, Berkeley, pp. 221–233.

Huber, P. (1974), *Robust Statistics*, Wiley, New York.

Imbens, G. W. (2002), 'Generalized method of moments and empirical likelihood', *Journal of Business and Economic Statistics* pp. 493–506.

Jöreskog, K. (1969), 'A general approach to confirmatory maximum likelihood factor analysis', *Psychometrika* **34**(2), 183–202.

Jöreskog, K. G. (1973), A general method for estimating a linear structural equation system, *in* A. Goldberger & O. Duncan, eds, 'Structural Equation Models in the Social Sciences', Seminar Press, New York, pp. 85–112.

Jöreskog, K. & Goldberger, A. (1972), 'Factor analysis by generalized least squares', *Psychometrika* **37**(3), 243–260.

Kendall, M., Stuart, A., Ord, K. J. & Arnold, S. (1999), *Advanced Theory of Statistics: Classical Inference and and the Linear Model*, Vol. 2A of *Kendall's Library of Statistics*, 6 edn, A Hodder Arnold Publication.

Kitamura, Y. (2007), Empirical likelihood methods in econometrics: Theory and practice, *in* R. Blundell, W. Newey & T. Persson, eds, 'Advances in Economics and Econometrics, Vol. III', Cambridge University Press, New York.

Kitamura, Y. & Stutzer, M. (1997), 'An information theoretic alternative to generalized method of moments estimation', *Econometrica* **65**(4), 861–874.

Lawley, D. N. (1940), 'The estimation of factor loadings by the method of maximum likelihood', *Proceedings of the Royal Society of Edinburgh* **60**, 64–82.

Lee, S. & Jennrich, R. (1979), 'A study of algorithms for covariance structure analysis with specific comparisons using factor analysis', *Psychometrika* **44**(1), 99–113.

MacCallum, R. C. & Austin, J. T. (2000), 'Applications of structural equation modeling in psychological research', *Annual Review of Psychology* **51**(1), 201–226.

Magnus, J. R. & Neudecker, H. (1999), *Matrix calculus with applications in statistics and econometrics*, 2nd edn, John Wiley & Sons.

Mardia, K. V. (1970), 'Measures of multivariate skewness and kurtosis with applications', *Biometrika* **57**, 519–530.

McArdle, J. J. & McDonald, R. P. (1984), 'Some algebraic properties of the reticular action model for moment structures.', *The British Journal of Mathematical and Statistical Psychology* **37**, 234–251.

Mittelhammer, R. C., Judge, G. G. & Miller, D. J. (2000), *Econometric Foundations*, Cambridge University Press.

Moustaki, I. & Victoria-Feser, M.-P. (2006), 'Bounded influence robust estimation in generalized linear latent variable models', *Journal of the American Statistical Association* **101**(474), 644–653. DOI 10.1198/016214505000001320.

Muthén, B. (1984), 'A general structural equation model with dichotomous, ordered categorical, and continuous latent variable indicators', *Psychometrika* **49**, 115–132.

Muthén, B. & Kaplan, D. (1985), 'A comparison of some methodologies for the factor analysis of non-normal Likert variables', *British Journal of Mathematical and Statistical Psychology* **38**(2), 171–189.

Muthén, B. & Kaplan, D. (1992), 'A comparison of some methodologies for the factor analysis of non-normal Likert variables: A note on the size of the model', *British Journal of Mathematical and Statistical Psychology* **45**(1), 19–30.

Neudecker, H. & Satorra, A. (1991), 'Linear structural relations: Gradient and Hessian of the fitting function', *Statistics and Probability Letters* **11**, 57–61.

Newey, W. K. & Smith, R. J. (2004), 'Higher order properties of gmm and generalized empirical likelihood estimators', *Econometrica* **72**(1), 219–255.

Owen, A. B. (1988), 'Empirical likelihood ratio confidence intervals for a single functional', *Biometrika* **75**(2), 237–249.

Owen, A. B. (1990), 'Empirical likelihood ratio confidence regions', *The Annals of Statistics* **18**(1), 90–120.

Owen, A. B. (1991), 'Empirical likelihood for linear models', *The Annals of Statistics* **19**(4), 1725–1747.

Owen, A. B. (2001), *Empirical Likelihood*, Chapman & Hall/CRC.

Pugesek, B. H., Tomer, A. & von Eye, A., eds (2002), *Structural Equation Modeling: Applications in Ecological and Evolutionary Biology*, Cambridge University Press.

Qin, J. & Lawless, J. (1994), 'Empirical likelihood and general estimating equations', *The Annals of Statistics* **22**(1), 300–325.

Sargan, J. D. (1958), 'The estimation of economic relationships using instrumental variables', *Econometrica* **26**(3), 393–415.

Satorra, A. (1990), 'Robustness issues in structural equation modeling: A review of recent developments', *Quality and Quantity* **24**, 367–386.

Satorra, A. & Bentler, P. M. (1990), 'Model conditions for asymptotic robustness in the analysis of linear relations', *Computational Statistics and Data Analysis* **10**(3), 235–249. doi:10.1016/0167-9473(90)90004-2.

Satorra, A. & Bentler, P. M. (1994), Corrections to test statistics and standard errors in covariance structure analysis, *in* A. von Eye & C. C. Clogg, eds, 'Latent variables analysis', Sage, Thousands Oaks, CA, pp. 399–419.

Siemsen, E. & Bollen, K. A. (2007), 'Least absolute deviation estimation in structural equation modeling', *Sociological Methods Research* **36**(2), 227–265.

Skrondal, A. (2000), 'Design and analysis of Monte Carlo experiments: Attacking the conventional wisdom', *Multivariate Behavioral Research* **35**(2), 137–167.

Smith, R. J. (1997), 'Alternative semi-parametric likelihood approaches to generalised method of moments estimation', *The Economic Journal* **107**(441), 503–519.

Stein, C. (1956), Efficient nonparametric testing and estimation, *in* 'Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability', Vol. 1, University of California Press, Berkeley.

Wald, A. (1948), 'Asymptotic properties of the maximum likelihood estimate of an unknown parameter of a discrete stochastic process', *The Annals of Mathematical Statistics* **19**(1), 40–46.

White, H. (1982), 'Maximum likelihood estimation of misspecified models', *Econometrica* **50**(1), 1–26.

Wilks, S. S. (1938), 'The large sample distribution of the likelihood ratio for testing composite hypotheses', *The Annals of Mathematical Statistics* **9**(1), 60–62.

Wooldridge, J. M. (2002), *Econometric Analysis of Cross Section and Panel Data*, MIT Press, Cambridge, MA.

Yuan, K.-H. & Bentler, P. M. (2007), Structural equation modeling, *in* C. Rao & S. Sinharay, eds, 'Handbook of Statistics: Psychometrics', Vol. 26 of *Handbook of Statistics*, Elsevier, chapter 10.

A Estimating equations for SEM estimators

For the maximum likelihood objective function (15), the matrix differential is

$$\begin{aligned}
\mathsf{d}\, F_{ML}(\theta, S) &= \mathsf{d}\{\log|\Sigma(\theta)| + \operatorname{tr}[\Sigma^{-1}(\theta)S] - \log|S| - \dim(x'y')'\} \\
&= \operatorname{tr}[\Sigma^{-1}(\theta)\,\mathsf{d}\,\Sigma(\theta)] - \operatorname{tr}[\Sigma^{-1}(\theta)\,\mathsf{d}\,\Sigma(\theta)\Sigma^{-1}S] \\
&= -\operatorname{tr}[\Sigma^{-1}(\theta)\,\mathsf{d}\,\Sigma(\theta)\Sigma^{-1}(S-\Sigma(\theta))] = -\operatorname{vec}[\mathsf{d}\,\Sigma'(\theta)]'(\Sigma^{-1}\otimes\Sigma^{-1})\operatorname{vec}[S-\Sigma(\theta)] \\
&= -\,\mathsf{d}\,\theta'\dot{\sigma}(\theta)'D'(\Sigma^{-1}\otimes\Sigma^{-1})D\operatorname{vech}[S-\Sigma(\theta)],
\end{aligned}$$

where D is duplication matrix (Magnus & Neudecker 1999), and $\dot{\sigma}(\cdot)$ is the Jacobian of the implied moments:

$$\dot{\sigma}(\theta) = \frac{\partial\sigma(\theta)}{\partial\theta'}.$$

In order to satisfy the first order condition

$$\mathsf{d}\,F_{ML}(\cdot) = 0$$

for arbitrary $\mathsf{d}\,\theta$, the estimates must satisfy

$$\dot{\sigma}(\theta)'D'(\Sigma^{-1}\otimes\Sigma^{-1})D(s-\sigma(\theta)) = 0.$$

These are the estimating equations for FIML estimation method.

Assuming the weight matrix W does not depend on the parameters over which minimization is performed, the differential of the WLS objective function (17) is

$$\mathsf{d}\,F_{WLS}(\theta, S, W) = \mathsf{d}\{[s-\sigma(\theta)]'W^{-1}[s-\sigma(\theta)]\} = 2\,\mathsf{d}\,\theta'\dot{\sigma}(\theta)'W^{-1}(s-\sigma(\theta)),$$

resulting in the first order conditions

$$\dot{\sigma}(\theta)'W^{-1}(s-\sigma(\theta)) = 0.$$

B Computation of EL estimates

Computing MELEs and profile likelihoods requires numeric optimization. Solvers capable of dealing with large number of variables are necessary for efficient computation. An interior point optimizing package "Knitro" for MATLAB was used in our early work and in the empirical example. Alternative optimizer NPSOL is much faster but often returns slightly violated constraints conditions which could make the point estimates and the value of χ^2 statistic inaccurate.

An alternative approach to finding the solutions $\tilde{\theta}$ and the empirical likelihood ratios $R(\theta)$ is based on duality of the optimization problem. The Lagrangian function for problem (3) is

$$H = \sum_{i=1}^{n}\log w_i + \lambda_0(1 - \sum_{i=1}^{n} w_i) - n\lambda'\sum_{i=1}^{n} w_i m(x_i, \theta), \qquad (37)$$

where dimension $t+q$ of the Lagrange multiplier λ matches that of $m(x_i, \theta)$. By the standard empirical likelihood calculations given in Owen (1990, 1991, 2001) or Qin & Lawless (1994), the Lagrange multiplier λ_0 can be shown to be equal to n, and eliminated, and the weights have explicit expression

$$w_i = \frac{1}{n} \frac{1}{1+\lambda' m(x_i, \theta)}. \tag{38}$$

Thus, $n+t-1$-dimensional problem (3) is reduced to a saddle point search over smaller $2t+q$ space:

$$\hat{\theta}_{EL} = \arg\max_{\theta} \min_{\lambda} \sum_{i=1}^{n} \log \frac{1}{n} \frac{1}{1+\lambda' m(x_i, \theta)}.$$

The solution is a point in the space of (θ, λ), so computations must be organized as an inner loop of minimization over λ, and an outer loop of maximization over θ (Hall & La Scala 1990, Kitamura 2007).

Also in order to obtain valid weights $0 \leq w_i \leq 1$, λ and θ must satisfy $1 + \lambda' m(x_i, \theta) \geq 1/n$ for each i. To bypass this complication, Owen (2001)[Sec. 3.14] suggested to modify the objective function. Introduce an augmented function

$$l_*(z) = \begin{cases} \log(z), & z \geq 1/n \\ \log(1/n) - 3/2 + 2nz - (nz)^2/2, & z < 1/n \end{cases}.$$

It coincides with $\log(z)$ in the valid domain, penalizes its argument for going below $1/n$ but unlike $\log(z)$ allows it to become negative. It also provides a smooth transition with two continuous derivatives at $z = 1/n$. Then the objective function that can be safely used without regard to any other remaining restrictions is

$$L_*(\lambda, \theta) = \sum_{i=1}^{n} l_*\big(1 + \lambda' m(x_i, \theta)\big).$$

The resulting estimates and test are asymptotically equivalent to the EL, since the effect of the modification is away from the relevant domain of the argument of $l_*(\cdot)$, which is $1 + O_p(n^{-1/2})$.

Another complication that may arise in the empirical likelihood computations is that the true value of the parameter is not feasible for any combinations of the empirical likelihood weights w_i unless the non-negativity condition is violated. In the simplest case of inference for the mean, the sample convex hull may fail to capture the true mean. For general estimating equations case, the situation is more complicated, as the set of parameter values supported by the sample is not convex, although generally connected. This problem is a convolution of a small sample and model specification issues. For properly specified models, the probability that zero is inside the convex hull of $m(x_i, \theta)$ goes to 1 exponentially fast as $n \to \infty$. Chen et al. (2008) proposed to augment the data, or estimating equations, by an extra artificial observation that is a scaled version of the sample mean. The "size" of this observation can be a slowly growing function of the sample size. Chen et al. (2008) suggested using

$$x_{n+1} = -a_n \bar{x}_n, \quad a_n = \max(1, \log(n)/2).$$

In the inference for the mean problems, they showed that this adjustment does not affect the asymptotic properties of EL, at the same time allowing to bypass the convex hull problem, and also incidentally improving upon undercoverage in small samples.

With MELE and EL ratio at hand, an overall model fit test can be performed directly with EL ratio value, and confidence sets can be explicitly written as

$$C_{\theta,\alpha} = \{\theta | 2\log\frac{R(\theta)}{R(\tilde{\theta})} \geq \chi^2_{u,\alpha}, \sum_{i=1}^{n} w_i \operatorname{vech}[Z_i Z_i' - \Sigma(\theta)] = 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1\}.$$

For individual confidence interval of θ_k, we can solve the following constrained programming problem:

$$\max \text{ or } \min\{\theta_k | 2\log\frac{R(\theta)}{R(\tilde{\theta})} \geq \chi^2_{1,\alpha}, \sum_{i=1}^{n} w_i \operatorname{vech}[(Z_i Z_i' - \Sigma(\theta))] = 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1\}.$$

to compute the maximum and minimum values allowed for specified significance level. Profile likelihood for parameter θ_k can be defined as

$$R^*(\theta_k; \theta_{-k}) = \max_w\{\prod_{i=1}^{n} nw_i | \sum_{i=1}^{n} w_i \operatorname{vech}[(Z_i Z_i' - \Sigma(\theta))] = 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1\},$$

in which all components of θ other than θ_k (denoted as θ_{-k}) are fixed. Then the confidence interval for θ_k is obtained by solving

$$-2\log R^*(\theta_k; \tilde{\theta}_{-k}) = \chi^2_{1,\alpha}, \tag{39}$$

for θ_k fixing the remaining components at their MELE values $\tilde{\theta}_{-k}$. The idea can be straightforwardly adapted for AEL, as well.

C Simulation results

The following tables summarize the simulation results. Four distributions were considered to study possible effects of skewness and kurtosis, either marginal or joint:

D1: $\xi \sim N(0,1)$, $\epsilon_k \sim N(0,1)$ for $k = 1,2,3$, ϵ_k is half-normal, scaled to have mean 0 and variance 1, with marginal skewness of $\sqrt{2}(4-\pi)/(\pi-2)^{3/2} = 0/407$ and kurtosis $8(\pi-3)/(\pi-2)^2 = 0.869$.

D2: $\xi \sim N(0,1)$, $\epsilon_k \sim N(0,1)$ for $k = 1,2,3$, ϵ_k is lognormal with mean of logs equal to $-0.5\ln(\mathrm{e}^1-1)-0.5$ and variances of the logs equal to 1, scaled to have mean 0 and variance 1. The marginal skewness of the error terms is $(\mathrm{e}+2)*\sqrt{(}\mathrm{e}-1) = 6.185$, and marginal kurtosis is $\mathrm{e}^4+2\mathrm{e}^3+3\mathrm{e}^2-6 = 110.9$.

D3: $\xi \sim N(0,1)$, $\epsilon_k \sim N(0,1)$ for $k = 1,2,3$, $(\epsilon_4,\epsilon_5,\epsilon_6)' \sim t_6$, multivariate t distribution, scaled to have unit marginal variances.

D4: x is multivariate skew normal (Azzalini & Dalla Valle 1996) with skewing vector 0.09(0,0,1,1,-1,-1), and the covariance matrix of the original distribution equal to

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2.0665 & 1.0665 & 0.9335 & 0.9335 \\ 1 & 1 & 1.0665 & 2.0665 & 0.9335 & 0.9335 \\ 1 & 1 & 0.9335 & 0.9335 & 2.0665 & 1.0665 \\ 1 & 1 & 0.9335 & 0.9335 & 1.0665 & 2.0665 \end{pmatrix}$$

Throughout this appendix, single star (*) indicates nominal significance at 1% level, and double star (**), at 0.1% level. F-tests are contrasts between estimators, for a given distribution, across all distributions. Significant p-value indicates that different estimators have different performance.

Table 2: Multivariate normality tests.

Sample size	Distribution **D1**	**D2**	**D3**	**D4**
	Multivariate skewness, $b_{1,p}$			
100	76.09	100.00	59.62	9.91
200	98.66	100.00	70.31	9.26
300	100.00	100.00	80.00	9.01
500	100.00	100.00	81.40	8.11
1000	100.00	100.00	93.70	9.42
	Multivariate kurtosis, $b_{2,p}$			
100	14.35	99.50	55.77	7.75
200	25.89	100.00	86.56	10.00
300	39.57	100.00	97.29	9.01
500	52.83	100.00	99.17	9.73
1000	82.96	100.00	100.00	9.96

Entries are Monte-Carlo estimates of the power of Mardia (1970) test of asymptotic 90% size.

Table 3: Asymptotic χ^2 approximation for goodness of fit tests.

	LM_{AEL}	LM_{EL}	T_{AEL}	T_{EL}	T_{FIML}	
D1, n= 100	0.000	0.000	0.000	0.000	0.458	K-S test p-value
	63.48	43.04	20.87	25.22	12.61	Empirical test size
200	0.000	0.000	0.008	0.001	0.386	K-S test p-value
	30.80	24.11	15.63	15.63	12.05	Empirical test size
300	0.000	0.003	0.258	0.115	0.281	K-S test p-value
	25.53	20.43	16.17	16.60	11.92	Empirical test size
500	0.000	0.068	0.540	0.365	0.730	K-S test p-value
	19.25	15.47	11.32	12.45	10.94	Empirical test size
1000	0.603	0.084	0.076	0.117	0.065	K-S test p-value
	13.00	9.87	8.97	8.97	6.73	Empirical test size
D2, n= 100	0.000	0.000	0.000	0.000	0.000	K-S test p-value
	83.50	74.04	33.84	40.12	13.40	Empirical test size
200	0.000	0.000	0.000	0.000	0.284	K-S test p-value
	62.54	55.63	24.74	27.65	12.37	Empirical test size
300	0.000	0.000	0.000	0.000	0.062	K-S test p-value
	55.18	49.02	22.50	23.78	11.12	Empirical test size
500	0.000	0.000	0.000	0.000	0.251	K-S test p-value
	40.49	38.13	18.24	18.76	9.60	Empirical test size
1000	0.000	0.000	0.000	0.000	0.813	K-S test p-value
	31.96	29.19	14.32	14.95	9.94	Empirical test size
D3, n= 100	0.000	0.000	0.000	0.000	0.000	K-S test p-value
	71.54	59.62	26.92	32.69	20.38	Empirical test size
200	0.000	0.000	0.001	0.000	0.000	K-S test p-value
	42.50	31.25	17.19	18.75	17.81	Empirical test size
300	0.000	0.000	0.001	0.000	0.000	K-S test p-value
	32.20	25.76	16.95	16.95	19.66	Empirical test size
500	0.000	0.027	0.453	0.714	0.003	K-S test p-value
	20.66	15.70	8.68	9.50	16.12	Empirical test size
1000	0.000	0.000	0.026	0.015	0.000	K-S test p-value
	19.33	17.65	15.13	15.13	21.43	Empirical test size
D4, n= 100	0.000	0.000	0.000	0.000	0.000	K-S test p-value
	73.51	58.74	32.25	37.12	21.98	Empirical test size
200	0.000	0.000	0.000	0.000	0.000	K-S test p-value
	63.33	51.85	37.22	39.82	32.04	Empirical test size
300	0.000	0.000	0.000	0.000	0.000	K-S test p-value
	65.44	58.09	47.98	49.45	44.12	Empirical test size
500	0.000	0.000	0.000	0.000	0.000	K-S test p-value
	70.45	66.67	60.18	60.90	58.92	Empirical test size
1000	0.000	0.000	0.000	0.000	0.000	K-S test p-value
	93.66	92.39	88.95	89.67	88.95	Empirical test size

Entries are p-values of Kolmogorov-Smirnov test statistic to compare the Monte Carlo distribution of a given test statistic with the asymptotic χ^2_9, and Monte Carlo rejection rates for the test of nominal 10% size using the quantile of the χ^2_9 distribution, $\text{Prob}[\chi^2_9 > 14.68] = 0.1$.

Table 4: Estimates bias.

The entries are regression estimates and their standard errors of the two-term expansion (34). The standard errors are corrected for clustering due to the same Monte Carlo sample.

	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, leading term for **D1**	-0.125	-0.113	-0.237	-0.154	-0.185	0.153
	(0.134)	(0.133)	(0.139)	(0.139)	(0.137)	(0.214)
MLE, leading term for **D1**	-0.082	-0.047	-0.205	-0.091	-0.136	0.073
	(0.130)	(0.128)	(0.135)	(0.135)	(0.131)	(0.206)
AEL, leading term for **D1**	-0.113	-0.079	-0.220	-0.159	-0.221	0.130
	(0.132)	(0.132)	(0.137)	(0.137)	(0.136)	(0.207)
EL, leading term for **D1**	-0.114	-0.077	-0.223	-0.158	-0.221	0.111
	(0.132)	(0.132)	(0.137)	(0.138)	(0.137)	(0.211)
ULS, leading term for **D2**	-0.034	-0.031	0.024	0.026	0.071	-0.086
	(0.059)	(0.060)	(0.061)	(0.061)	(0.060)	(0.097)
MLE, leading term for **D2**	-0.051	-0.030	0.011	0.008	0.036	-0.041
	(0.057)	(0.058)	(0.059)	(0.059)	(0.058)	(0.095)
AEL, leading term for **D2**	-0.060	-0.049	-0.069	-0.059	-0.072	0.076
	(0.060)	(0.060)	(0.061)	(0.059)	(0.059)	(0.096)
EL, leading term for **D2**	-0.057	-0.046	-0.063	-0.055	-0.071	-0.015
	(0.060)	(0.060)	(0.061)	(0.059)	(0.059)	(0.098)
ULS, leading term for **D3**	-0.074	0.014	0.050	0.036	-0.039	-0.044
	(0.130)	(0.134)	(0.132)	(0.130)	(0.129)	(0.209)
MLE, leading term for **D3**	-0.078	0.006	0.076	0.052	-0.045	-0.064
	(0.128)	(0.132)	(0.129)	(0.128)	(0.128)	(0.205)
AEL, leading term for **D3**	-0.066	0.030	0.118	0.075	0.007	-0.111
	(0.131)	(0.134)	(0.129)	(0.129)	(0.132)	(0.207)
EL, leading term for **D3**	-0.065	0.032	0.118	0.076	0.005	-0.157
	(0.131)	(0.134)	(0.129)	(0.129)	(0.132)	(0.212)
ULS, leading term for **D4**	-0.051	-0.659**	-0.670**	-0.659**	-0.547**	0.422*
	(0.089)	(0.092)	(0.092)	(0.089)	(0.089)	(0.146)
MLE, leading term for **D4**	-0.053	-0.653**	-0.669**	-0.665**	-0.536**	0.435*
	(0.086)	(0.089)	(0.089)	(0.086)	(0.087)	(0.141)
AEL, leading term for **D4**	-0.036	-0.673**	-0.653**	-0.689**	-0.534**	0.430*
	(0.090)	(0.093)	(0.093)	(0.091)	(0.090)	(0.142)
EL, leading term for **D4**	-0.035	-0.673**	-0.656**	-0.688**	-0.536**	0.400*
	(0.090)	(0.093)	(0.093)	(0.091)	(0.090)	(0.145)

	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, $n^{-1/2}$ term for **D1**	3.837	2.452	4.639	3.485	3.436	-1.738
	(2.140)	(2.075)	(2.212)	(2.207)	(2.147)	(3.332)
MLE, $n^{-1/2}$ term for **D1**	3.157	1.363	4.098	2.197	2.226	-0.470
	(2.035)	(1.961)	(2.129)	(2.118)	(2.033)	(3.174)
AEL, $n^{-1/2}$ term for **D1**	3.938	2.049	4.379	3.315	3.645	-5.529
	(2.097)	(2.088)	(2.212)	(2.201)	(2.205)	(3.203)
EL, $n^{-1/2}$ term for **D1**	3.966	2.005	4.428	3.296	3.656	-0.753
	(2.097)	(2.095)	(2.211)	(2.212)	(2.211)	(3.307)
ULS, $n^{-1/2}$ term for **D2**	1.654	1.744	0.654	1.285	0.038	2.186
	(0.913)	(0.942)	(0.970)	(0.950)	(0.937)	(1.506)
MLE, $n^{-1/2}$ term for **D2**	1.954	1.558	0.930	1.579	0.703	1.337
	(0.873)	(0.901)	(0.932)	(0.898)	(0.910)	(1.438)
AEL, $n^{-1/2}$ term for **D2**	2.174	1.806	1.416	1.786	1.577	-4.400*
	(0.933)	(0.957)	(0.960)	(0.923)	(0.919)	(1.462)
EL, $n^{-1/2}$ term for **D2**	2.118	1.740	1.297	1.699	1.554	2.004
	(0.934)	(0.958)	(0.961)	(0.922)	(0.922)	(1.521)
ULS, $n^{-1/2}$ term for **D3**	3.516	1.934	0.316	0.679	2.657	-0.322
	(2.037)	(2.080)	(2.045)	(2.014)	(2.025)	(3.127)
MLE, $n^{-1/2}$ term for **D3**	3.119	1.547	0.146	0.518	2.564	0.045
	(1.962)	(2.014)	(1.967)	(1.942)	(1.996)	(3.030)
AEL, $n^{-1/2}$ term for **D3**	3.116	1.033	-0.257	0.130	1.730	-3.230
	(2.047)	(2.055)	(1.980)	(1.993)	(2.109)	(3.060)
EL, $n^{-1/2}$ term for **D3**	3.085	0.992	-0.264	0.120	1.764	2.060
	(2.048)	(2.058)	(1.982)	(1.995)	(2.114)	(3.183)
ULS, $n^{-1/2}$ term for **D4**	2.160	7.778**	7.576**	6.305**	5.723**	-3.029
	(1.432)	(1.453)	(1.472)	(1.397)	(1.430)	(2.307)
MLE, $n^{-1/2}$ term for **D4**	2.397	7.929**	7.902**	6.718**	5.652**	-3.567
	(1.349)	(1.387)	(1.397)	(1.321)	(1.352)	(2.183)
AEL, $n^{-1/2}$ term for **D4**	2.396	8.639**	7.879**	7.757**	6.008**	-8.580**
	(1.426)	(1.469)	(1.486)	(1.434)	(1.425)	(2.209)
EL, $n^{-1/2}$ term for **D4**	2.372	8.629**	7.935**	7.737**	6.033**	-3.219
	(1.426)	(1.471)	(1.483)	(1.434)	(1.429)	(2.282)
F-test, leading terms	0.602	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.126	0.000	0.000	0.000	0.000	0.000

	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, leading term for **D1**	-0.298	-0.139	-0.267	0.049	0.074	0.056
	(0.148)	(0.140)	(0.138)	(0.165)	(0.171)	(0.163)
MLE, leading term for **D1**	-0.270	-0.181	-0.275	0.025	0.022	0.064
	(0.142)	(0.134)	(0.132)	(0.159)	(0.164)	(0.157)
AEL, leading term for **D1**	-0.119	-0.144	-0.169	0.209	0.220	0.253
	(0.144)	(0.138)	(0.131)	(0.161)	(0.169)	(0.157)
EL, leading term for **D1**	-0.145	-0.159	-0.194	0.186	0.211	0.240
	(0.146)	(0.140)	(0.133)	(0.164)	(0.171)	(0.160)
ULS, leading term for **D2**	-0.047	0.008	-0.028	-0.364	-0.323	-0.192
	(0.065)	(0.063)	(0.064)	(0.350)	(0.307)	(0.320)
MLE, leading term for **D2**	-0.056	-0.010	-0.021	-0.257	-0.305	-0.260
	(0.063)	(0.061)	(0.062)	(0.347)	(0.299)	(0.317)
AEL, leading term for **D2**	0.067	0.110	0.155	-1.282**	-1.419**	-1.103**
	(0.063)	(0.062)	(0.062)	(0.255)	(0.224)	(0.247)
EL, leading term for **D2**	-0.005	0.038	0.078	-1.292**	-1.434**	-1.082**
	(0.064)	(0.063)	(0.063)	(0.258)	(0.226)	(0.247)
ULS, leading term for **D3**	0.277	-0.026	0.049	-0.261	-0.160	0.189
	(0.138)	(0.138)	(0.140)	(0.203)	(0.197)	(0.218)
MLE, leading term for **D3**	0.269	-0.032	0.069	-0.277	-0.098	0.194
	(0.135)	(0.136)	(0.137)	(0.197)	(0.192)	(0.213)
AEL, leading term for **D3**	0.445*	0.130	0.242	-0.199	-0.116	0.299
	(0.136)	(0.137)	(0.135)	(0.169)	(0.183)	(0.193)
EL, leading term for **D3**	0.421*	0.098	0.209	-0.228	-0.146	0.279
	(0.138)	(0.139)	(0.137)	(0.171)	(0.186)	(0.196)
ULS, leading term for **D4**	-0.417**	-0.454**	3.046**	3.087**	2.965**	3.055**
	(0.093)	(0.094)	(0.099)	(0.099)	(0.106)	(0.101)
MLE, leading term for **D4**	-0.397**	-0.435**	2.965**	3.019**	2.951**	3.043**
	(0.090)	(0.092)	(0.096)	(0.096)	(0.102)	(0.099)
AEL, leading term for **D4**	-0.226	-0.271*	3.143**	3.133**	3.111**	3.178**
	(0.092)	(0.093)	(0.099)	(0.097)	(0.105)	(0.099)
EL, leading term for **D4**	-0.251*	-0.300*	3.116**	3.103**	3.081**	3.150**
	(0.093)	(0.094)	(0.100)	(0.099)	(0.106)	(0.101)

	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, $n^{-1/2}$ term for **D1**	2.863	-1.744	1.648	-1.917	-4.320	-3.152
	(2.329)	(2.166)	(2.168)	(2.575)	(2.597)	(2.561)
MLE, $n^{-1/2}$ term for **D1**	2.190	-0.939	1.691	-1.697	-3.484	-3.311
	(2.215)	(2.052)	(2.057)	(2.468)	(2.485)	(2.458)
AEL, $n^{-1/2}$ term for **D1**	-6.165*	-7.451**	-6.232*	-10.648**	-13.222**	-13.041**
	(2.243)	(2.160)	(2.056)	(2.498)	(2.582)	(2.478)
EL, $n^{-1/2}$ term for **D1**	-1.244	-2.788	-1.372	-5.766	-8.672*	-8.397**
	(2.310)	(2.210)	(2.100)	(2.576)	(2.636)	(2.541)
ULS, $n^{-1/2}$ term for **D2**	-1.747	-2.070	-1.257	3.022	2.148	1.080
	(0.998)	(0.973)	(1.001)	(5.664)	(4.756)	(4.769)
MLE, $n^{-1/2}$ term for **D2**	-1.574	-1.637	-1.363	1.216	1.452	2.458
	(0.940)	(0.929)	(0.947)	(5.559)	(4.500)	(4.701)
AEL, $n^{-1/2}$ term for **D2**	-10.298**	-10.523**	-10.879**	-18.433**	-18.036**	-22.220**
	(0.956)	(0.937)	(0.949)	(3.593)	(3.029)	(3.337)
EL, $n^{-1/2}$ term for **D2**	-4.329**	-4.529**	-4.789**	-13.922**	-13.401**	-18.327**
	(0.979)	(0.959)	(0.973)	(3.669)	(3.089)	(3.297)
ULS, $n^{-1/2}$ term for **D3**	-6.675*	-2.079	-2.046	2.136	0.622	-6.212
	(2.080)	(2.141)	(2.114)	(3.313)	(3.006)	(3.307)
MLE, $n^{-1/2}$ term for **D3**	-6.329*	-1.247	-2.249	2.130	-0.876	-6.442
	(2.004)	(2.074)	(2.045)	(3.118)	(2.893)	(3.189)
AEL, $n^{-1/2}$ term for **D3**	-15.227**	-10.030**	-10.869**	-11.035**	-12.767**	-19.753**
	(2.036)	(2.093)	(2.014)	(2.521)	(2.691)	(2.785)
EL, $n^{-1/2}$ term for **D3**	-10.362**	-4.989	-5.813*	-6.078	-7.820*	-14.986**
	(2.086)	(2.147)	(2.058)	(2.580)	(2.763)	(2.865)
ULS, $n^{-1/2}$ term for **D4**	1.112	2.054	-27.242**	-27.259**	-24.438**	-26.475**
	(1.458)	(1.470)	(1.552)	(1.553)	(1.688)	(1.582)
MLE, $n^{-1/2}$ term for **D4**	0.850	1.961	-25.776**	-26.220**	-24.079**	-25.923**
	(1.381)	(1.401)	(1.476)	(1.456)	(1.568)	(1.503)
AEL, $n^{-1/2}$ term for **D4**	-8.364**	-6.912**	-36.605**	-35.525**	-34.699**	-36.304**
	(1.415)	(1.425)	(1.514)	(1.482)	(1.610)	(1.513)
EL, $n^{-1/2}$ term for **D4**	-3.236	-1.690	-30.927**	-29.781**	-28.928**	-30.595**
	(1.447)	(1.463)	(1.550)	(1.523)	(1.648)	(1.553)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

Table 5: Estimates efficiency.

The entries are regression estimates and their standard errors of the two-term expansion (35). The standard errors are corrected for clustering due to the same Monte Carlo sample.

	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, leading term for **D1**	1.973	2.317	2.313	2.340	2.340	6.236
	(0.317)	(0.304)	(0.420)	(0.392)	(0.346)	(0.747)
MLE, leading term for **D1**	2.170	2.472	2.282	2.305	2.393	6.468
	(0.306)	(0.291)	(0.396)	(0.382)	(0.333)	(0.732)
AEL, leading term for **D1**	2.109	2.139	1.977	2.059	1.786	6.233
	(0.351)	(0.323)	(0.455)	(0.407)	(0.408)	(0.719)
EL, leading term for **D1**	2.109	2.118	1.980	2.024	1.767	5.927
	(0.351)	(0.324)	(0.457)	(0.409)	(0.409)	(0.762)
ULS, leading term for **D2**	2.411	2.228	2.130	2.367	2.280	6.641
	(0.139)	(0.151)	(0.159)	(0.167)	(0.145)	(0.368)
MLE, leading term for **D2**	2.408	2.188	2.059	2.429	2.153	6.670
	(0.139)	(0.153)	(0.155)	(0.151)	(0.150)	(0.355)
AEL, leading term for **D2**	2.121	1.869	1.877	2.123	2.121	6.472
	(0.161)	(0.171)	(0.205)	(0.155)	(0.155)	(0.360)
EL, leading term for **D2**	2.113	1.862	1.874	2.136	2.104	6.012
	(0.162)	(0.172)	(0.208)	(0.154)	(0.156)	(0.382)
ULS, leading term for **D3**	2.194	2.464	2.388	2.475	2.142	7.726
	(0.306)	(0.341)	(0.331)	(0.303)	(0.281)	(0.783)
MLE, leading term for **D3**	2.256	2.481	2.410	2.540	2.045	7.756
	(0.300)	(0.337)	(0.317)	(0.291)	(0.284)	(0.764)
AEL, leading term for **D3**	1.987	2.455	2.332	2.331	1.602	7.733
	(0.318)	(0.318)	(0.323)	(0.300)	(0.335)	(0.801)
EL, leading term for **D3**	1.984	2.442	2.325	2.324	1.587	7.292
	(0.319)	(0.318)	(0.323)	(0.302)	(0.338)	(0.868)
ULS, leading term for **D4**	2.197	2.577	2.337	2.552	2.260	6.439
	(0.228)	(0.236)	(0.233)	(0.212)	(0.229)	(0.589)
MLE, leading term for **D4**	2.209	2.517	2.264	2.545	2.252	6.335
	(0.216)	(0.233)	(0.224)	(0.207)	(0.215)	(0.559)
AEL, leading term for **D4**	2.104	2.338	2.173	2.320	2.192	6.441
	(0.238)	(0.240)	(0.240)	(0.249)	(0.235)	(0.565)
EL, leading term for **D4**	2.105	2.329	2.189	2.325	2.171	6.084
	(0.238)	(0.241)	(0.239)	(0.248)	(0.236)	(0.593)

	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, $n^{-1/2}$ term for **D1**	7.698	1.203	4.745	5.485	1.590	2.611
	(5.656)	(4.971)	(7.641)	(6.855)	(5.846)	(11.828)
MLE, $n^{-1/2}$ term for **D1**	5.000	-1.202	5.985	6.280	0.674	-0.218
	(5.285)	(4.663)	(7.101)	(6.711)	(5.501)	(11.613)
AEL, $n^{-1/2}$ term for **D1**	8.606	7.241	13.567	12.437	14.756	4.686
	(6.295)	(5.515)	(8.617)	(7.375)	(7.535)	(11.350)
EL, $n^{-1/2}$ term for **D1**	8.612	7.708	13.498	13.187	15.156	14.112
	(6.299)	(5.543)	(8.654)	(7.443)	(7.582)	(12.449)
ULS, $n^{-1/2}$ term for **D2**	1.639	5.997	8.386	4.278	5.134	1.853
	(2.256)	(2.547)	(2.816)	(2.887)	(2.415)	(5.818)
MLE, $n^{-1/2}$ term for **D2**	1.089	5.591	9.220	2.464	7.277	0.288
	(2.256)	(2.627)	(2.714)	(2.467)	(2.557)	(5.473)
AEL, $n^{-1/2}$ term for **D2**	8.709	13.708	12.659	7.630	7.209	5.329
	(2.815)	(3.083)	(3.840)	(2.616)	(2.686)	(5.619)
EL, $n^{-1/2}$ term for **D2**	8.920	13.904	12.776	7.420	7.642	18.300
	(2.831)	(3.089)	(3.908)	(2.591)	(2.705)	(6.170)
ULS, $n^{-1/2}$ term for **D3**	5.620	3.265	3.089	3.361	7.205	-20.386
	(5.133)	(5.909)	(5.418)	(5.019)	(4.576)	(12.171)
MLE, $n^{-1/2}$ term for **D3**	3.775	2.524	2.353	2.004	9.122	-22.902
	(4.904)	(5.717)	(5.001)	(4.697)	(4.594)	(11.658)
AEL, $n^{-1/2}$ term for **D3**	10.424	4.350	4.454	6.816	18.582	-20.512
	(5.403)	(5.126)	(5.111)	(4.917)	(5.897)	(12.581)
EL, $n^{-1/2}$ term for **D3**	10.511	4.616	4.617	6.992	18.909	-8.394
	(5.420)	(5.139)	(5.115)	(4.948)	(5.962)	(14.256)
ULS, $n^{-1/2}$ term for **D4**	6.286	1.737	4.542	0.107	5.096	4.480
	(3.852)	(3.982)	(3.983)	(3.363)	(3.944)	(10.014)
MLE, $n^{-1/2}$ term for **D4**	6.428	3.423	6.496	0.692	5.798	7.296
	(3.506)	(3.887)	(3.733)	(3.232)	(3.575)	(9.202)
AEL, $n^{-1/2}$ term for **D4**	12.466	10.613	13.305	9.928	11.517	9.518
	(4.047)	(4.044)	(4.038)	(4.207)	(4.025)	(9.281)
EL, $n^{-1/2}$ term for **D4**	12.466	10.812	12.961	9.834	11.985	20.442
	(4.040)	(4.053)	(4.011)	(4.164)	(4.048)	(9.962)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, leading term for **D1**	2.876	2.848	2.569	3.766	4.628	3.514
	(0.371)	(0.319)	(0.327)	(0.470)	(0.490)	(0.505)
MLE, leading term for **D1**	2.878	2.840	2.504	3.725	4.511	3.325
	(0.359)	(0.311)	(0.309)	(0.455)	(0.474)	(0.481)
AEL, leading term for **D1**	2.865	2.449	2.420	3.713	4.383	3.132
	(0.382)	(0.340)	(0.300)	(0.484)	(0.498)	(0.511)
EL, leading term for **D1**	2.731	2.372	2.369	3.545	4.307	3.010
	(0.399)	(0.346)	(0.308)	(0.507)	(0.513)	(0.542)
ULS, leading term for **D2**	3.064	2.952	2.727	57.571	70.266	95.808
	(0.157)	(0.155)	(0.158)	(23.595)	(9.970)	(12.500)
MLE, leading term for **D2**	3.113	2.893	2.762	54.112	73.133	91.842
	(0.151)	(0.151)	(0.152)	(30.146)	(9.534)	(13.071)
AEL, leading term for **D2**	3.091	2.862	2.790	67.301	60.929	75.276
	(0.154)	(0.157)	(0.152)	(9.995)	(3.463)	(9.668)
EL, leading term for **D2**	3.013	2.794	2.701	66.971	61.051	78.215
	(0.159)	(0.163)	(0.159)	(10.450)	(3.552)	(9.460)
ULS, leading term for **D3**	3.141	2.779	3.283	3.437	6.077	7.595
	(0.328)	(0.369)	(0.363)	(1.453)	(0.858)	(1.904)
MLE, leading term for **D3**	3.201	2.821	3.276	3.870	6.115	7.655
	(0.317)	(0.369)	(0.358)	(1.344)	(0.805)	(1.900)
AEL, leading term for **D3**	3.198	2.882	3.351	5.066	6.352	7.505
	(0.308)	(0.350)	(0.358)	(0.522)	(0.697)	(1.054)
EL, leading term for **D3**	3.120	2.780	3.293	4.967	6.191	7.322
	(0.318)	(0.370)	(0.369)	(0.538)	(0.713)	(1.083)
ULS, leading term for **D4**	2.856	3.020	3.140	3.312	3.288	3.351
	(0.220)	(0.243)	(0.248)	(0.252)	(0.290)	(0.258)
MLE, leading term for **D4**	2.857	2.966	3.173	3.462	3.466	3.379
	(0.215)	(0.239)	(0.242)	(0.245)	(0.275)	(0.258)
AEL, leading term for **D4**	2.836	2.995	3.337	3.518	3.636	3.497
	(0.225)	(0.242)	(0.256)	(0.255)	(0.308)	(0.272)
EL, leading term for **D4**	2.769	2.893	3.268	3.431	3.550	3.398
	(0.232)	(0.250)	(0.264)	(0.266)	(0.319)	(0.283)

	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, $n^{-1/2}$ term for **D1**	4.650	-2.900	5.049	1.636	-11.425	3.180
	(5.814)	(4.843)	(5.094)	(7.632)	(7.197)	(8.851)
MLE, $n^{-1/2}$ term for **D1**	4.429	-2.607	5.920	2.862	-9.156	6.277
	(5.614)	(4.731)	(4.761)	(7.313)	(6.937)	(8.422)
AEL, $n^{-1/2}$ term for **D1**	5.894	5.990	7.458	5.172	-4.240	10.827
	(6.128)	(5.684)	(4.658)	(8.037)	(7.524)	(9.527)
EL, $n^{-1/2}$ term for **D1**	10.192	8.891	9.916	10.590	-0.765	15.158
	(6.545)	(5.810)	(4.841)	(8.600)	(7.856)	(10.259)
ULS, $n^{-1/2}$ term for **D2**	-2.211	-1.887	3.305	418.918	52.535	-191.131
	(2.349)	(2.338)	(2.511)	(456.246)	(173.799)	(189.255)
MLE, $n^{-1/2}$ term for **D2**	-4.116	-1.603	1.594	515.605	-15.057	-115.106
	(2.227)	(2.251)	(2.344)	(603.379)	(160.489)	(199.914)
AEL, $n^{-1/2}$ term for **D2**	-2.837	-0.864	1.060	-433.131	-455.161	-569.998
	(2.306)	(2.431)	(2.326)	(150.233)	(45.925)	(122.664)
EL, $n^{-1/2}$ term for **D2**	0.209	1.966	4.295	-404.847	-439.127	-609.404
	(2.416)	(2.570)	(2.504)	(161.682)	(47.809)	(111.105)
ULS, $n^{-1/2}$ term for **D3**	-4.868	4.474	-3.059	43.845	-3.866	-13.443
	(4.928)	(6.290)	(5.325)	(28.944)	(13.054)	(27.874)
MLE, $n^{-1/2}$ term for **D3**	-5.736	3.088	-3.798	34.223	-6.211	-17.643
	(4.692)	(6.175)	(5.199)	(25.754)	(12.069)	(27.043)
AEL, $n^{-1/2}$ term for **D3**	-4.250	3.471	-5.603	-5.635	-20.490	-36.826
	(4.430)	(5.633)	(5.238)	(7.635)	(9.835)	(14.502)
EL, $n^{-1/2}$ term for **D3**	-1.215	7.050	-2.955	-1.142	-14.730	-30.509
	(4.643)	(6.088)	(5.482)	(8.005)	(10.201)	(15.193)
ULS, $n^{-1/2}$ term for **D4**	-0.781	-1.974	0.219	-0.996	3.381	-2.742
	(3.402)	(3.942)	(3.954)	(3.932)	(4.851)	(4.068)
MLE, $n^{-1/2}$ term for **D4**	-0.024	-0.439	0.642	-2.925	0.747	-1.945
	(3.252)	(3.808)	(3.727)	(3.697)	(4.400)	(3.961)
AEL, $n^{-1/2}$ term for **D4**	1.922	0.727	0.995	-1.192	1.101	-1.950
	(3.578)	(3.871)	(4.056)	(3.803)	(5.036)	(4.243)
EL, $n^{-1/2}$ term for **D4**	4.797	4.340	4.149	2.482	4.729	1.723
	(3.751)	(4.061)	(4.244)	(4.103)	(5.284)	(4.516)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

Table 6: Estimates mean squared error.

The entries are regression estimates and their standard errors of the two-term expansion accurate to $o_p(n^{-1})$. The standard errors are corrected for clustering due to the same Monte Carlo sample.

	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, leading term for **D1**	1.918	2.282	2.190	2.274	2.272	6.247
	(0.350)	(0.312)	(0.451)	(0.414)	(0.365)	(0.737)
MLE, leading term for **D1**	2.127	2.455	2.187	2.282	2.365	6.494
	(0.329)	(0.297)	(0.421)	(0.397)	(0.343)	(0.723)
AEL, leading term for **D1**	2.038	2.114	1.875	2.014	1.737	6.039
	(0.381)	(0.330)	(0.484)	(0.424)	(0.426)	(0.697)
EL, leading term for **D1**	2.036	2.093	1.877	1.980	1.718	5.952
	(0.381)	(0.331)	(0.486)	(0.426)	(0.428)	(0.752)
ULS, leading term for **D2**	2.401	2.226	2.134	2.365	2.286	6.632
	(0.142)	(0.153)	(0.161)	(0.170)	(0.146)	(0.370)
MLE, leading term for **D2**	2.393	2.187	2.060	2.422	2.154	6.668
	(0.142)	(0.155)	(0.158)	(0.153)	(0.152)	(0.356)
AEL, leading term for **D2**	2.100	1.862	1.871	2.114	2.113	6.377
	(0.166)	(0.175)	(0.207)	(0.157)	(0.157)	(0.355)
EL, leading term for **D2**	2.093	1.855	1.870	2.128	2.096	6.004
	(0.167)	(0.175)	(0.210)	(0.156)	(0.157)	(0.386)
ULS, leading term for **D3**	2.117	2.443	2.390	2.474	2.114	7.738
	(0.329)	(0.354)	(0.336)	(0.307)	(0.286)	(0.776)
MLE, leading term for **D3**	2.198	2.469	2.418	2.545	2.024	7.785
	(0.320)	(0.350)	(0.322)	(0.295)	(0.288)	(0.760)
AEL, leading term for **D3**	1.932	2.457	2.351	2.344	1.599	7.706
	(0.338)	(0.327)	(0.329)	(0.306)	(0.342)	(0.789)
EL, leading term for **D3**	1.930	2.445	2.344	2.336	1.584	7.358
	(0.338)	(0.328)	(0.329)	(0.307)	(0.345)	(0.869)
ULS, leading term for **D4**	2.180	2.822	2.591	2.834	2.478	6.620
	(0.233)	(0.244)	(0.241)	(0.219)	(0.234)	(0.606)
MLE, leading term for **D4**	2.188	2.735	2.487	2.803	2.454	6.506
	(0.221)	(0.241)	(0.232)	(0.214)	(0.220)	(0.571)
AEL, leading term for **D4**	2.083	2.524	2.378	2.548	2.377	6.367
	(0.243)	(0.250)	(0.247)	(0.257)	(0.240)	(0.557)
EL, leading term for **D4**	2.084	2.516	2.395	2.554	2.357	6.229
	(0.243)	(0.250)	(0.246)	(0.256)	(0.241)	(0.604)

	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, $n^{-1/2}$ term for **D1**	9.597	2.008	7.530	7.271	3.193	3.310
	(6.346)	(5.184)	(8.398)	(7.359)	(6.318)	(11.569)
MLE, $n^{-1/2}$ term for **D1**	6.439	-0.777	8.169	7.107	1.399	0.028
	(5.786)	(4.801)	(7.707)	(7.067)	(5.752)	(11.400)
AEL, $n^{-1/2}$ term for **D1**	10.620	7.824	15.862	13.699	15.946	9.583
	(6.991)	(5.691)	(9.297)	(7.762)	(7.979)	(10.816)
EL, $n^{-1/2}$ term for **D1**	10.649	8.273	15.830	14.436	16.351	14.435
	(6.996)	(5.718)	(9.340)	(7.819)	(8.033)	(12.198)
ULS, $n^{-1/2}$ term for **D2**	1.906	6.186	8.423	4.522	5.134	2.092
	(2.321)	(2.591)	(2.854)	(2.932)	(2.437)	(5.891)
MLE, $n^{-1/2}$ term for **D2**	1.450	5.731	9.305	2.787	7.374	0.380
	(2.331)	(2.671)	(2.765)	(2.505)	(2.597)	(5.517)
AEL, $n^{-1/2}$ term for **D2**	9.199	13.924	12.799	7.852	7.391	7.717
	(2.930)	(3.158)	(3.890)	(2.680)	(2.723)	(5.494)
EL, $n^{-1/2}$ term for **D2**	9.389	14.104	12.893	7.622	7.821	18.665
	(2.945)	(3.160)	(3.953)	(2.649)	(2.741)	(6.275)
ULS, $n^{-1/2}$ term for **D3**	7.503	4.002	3.165	3.492	8.043	-20.329
	(5.680)	(6.179)	(5.499)	(5.126)	(4.714)	(11.992)
MLE, $n^{-1/2}$ term for **D3**	5.125	3.011	2.366	2.093	9.852	-23.129
	(5.357)	(5.970)	(5.080)	(4.786)	(4.717)	(11.538)
AEL, $n^{-1/2}$ term for **D3**	11.751	4.586	4.357	6.803	19.033	-18.100
	(5.837)	(5.292)	(5.219)	(5.025)	(6.055)	(12.244)
EL, $n^{-1/2}$ term for **D3**	11.817	4.836	4.519	6.977	19.370	-9.113
	(5.854)	(5.305)	(5.222)	(5.056)	(6.125)	(14.242)
ULS, $n^{-1/2}$ term for **D4**	6.724	-1.088	1.658	-2.917	2.508	2.635
	(3.966)	(4.118)	(4.109)	(3.433)	(3.991)	(10.282)
MLE, $n^{-1/2}$ term for **D4**	6.977	1.001	4.069	-2.040	3.436	5.516
	(3.621)	(4.031)	(3.874)	(3.314)	(3.626)	(9.373)
AEL, $n^{-1/2}$ term for **D4**	13.082	8.718	11.045	7.526	9.331	11.651
	(4.156)	(4.243)	(4.173)	(4.341)	(4.096)	(9.006)
EL, $n^{-1/2}$ term for **D4**	13.072	8.905	10.706	7.418	9.794	18.948
	(4.146)	(4.251)	(4.152)	(4.296)	(4.120)	(10.129)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, leading term for **D1**	2.926	2.853	2.618	3.774	4.529	3.477
	(0.369)	(0.334)	(0.332)	(0.468)	(0.488)	(0.490)
MLE, leading term for **D1**	2.943	2.870	2.559	3.748	4.457	3.282
	(0.356)	(0.322)	(0.313)	(0.453)	(0.470)	(0.467)
AEL, leading term for **D1**	2.715	2.222	2.303	3.328	3.671	2.463
	(0.384)	(0.375)	(0.328)	(0.484)	(0.508)	(0.472)
EL, leading term for **D1**	2.754	2.361	2.402	3.481	4.014	2.757
	(0.391)	(0.354)	(0.314)	(0.499)	(0.502)	(0.504)
ULS, leading term for **D2**	3.052	2.931	2.728	57.666	70.423	95.872
	(0.156)	(0.155)	(0.158)	(23.573)	(9.851)	(12.398)
MLE, leading term for **D2**	3.105	2.880	2.760	54.192	73.291	91.938
	(0.151)	(0.150)	(0.151)	(30.105)	(9.411)	(12.982)
AEL, leading term for **D2**	2.626	2.370	2.315	67.867	61.699	74.763
	(0.167)	(0.163)	(0.164)	(9.398)	(2.968)	(9.117)
EL, leading term for **D2**	2.929	2.698	2.616	68.101	62.453	78.241
	(0.160)	(0.160)	(0.160)	(9.897)	(3.081)	(8.994)
ULS, leading term for **D3**	3.071	2.754	3.261	3.489	6.120	7.555
	(0.328)	(0.364)	(0.360)	(1.448)	(0.840)	(1.880)
MLE, leading term for **D3**	3.138	2.806	3.247	3.925	6.146	7.605
	(0.317)	(0.365)	(0.355)	(1.338)	(0.783)	(1.875)
AEL, leading term for **D3**	2.406	2.407	2.813	4.581	5.718	6.100
	(0.344)	(0.366)	(0.365)	(0.582)	(0.680)	(1.014)
EL, leading term for **D3**	2.843	2.652	3.142	4.862	5.994	6.576
	(0.327)	(0.367)	(0.364)	(0.559)	(0.682)	(1.046)
ULS, leading term for **D4**	3.033	3.235	10.311	10.797	10.503	10.746
	(0.223)	(0.241)	(0.452)	(0.478)	(0.518)	(0.482)
MLE, leading term for **D4**	3.020	3.163	10.075	10.681	10.554	10.695
	(0.217)	(0.237)	(0.438)	(0.462)	(0.494)	(0.474)
AEL, leading term for **D4**	2.572	2.805	8.804	9.305	9.331	9.256
	(0.250)	(0.256)	(0.418)	(0.431)	(0.481)	(0.437)
EL, leading term for **D4**	2.787	2.962	9.996	10.419	10.435	10.405
	(0.238)	(0.249)	(0.451)	(0.469)	(0.525)	(0.476)

	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, $n^{-1/2}$ term for **D1**	4.190	-1.831	4.831	1.761	-8.649	4.208
	(5.826)	(5.023)	(5.145)	(7.577)	(7.147)	(8.461)
MLE, $n^{-1/2}$ term for **D1**	3.762	-2.107	5.612	2.808	-7.246	7.424
	(5.572)	(4.843)	(4.800)	(7.263)	(6.837)	(8.033)
AEL, $n^{-1/2}$ term for **D1**	12.800	16.168	14.716	15.910	14.949	28.211
	(6.195)	(6.246)	(5.219)	(7.938)	(7.769)	(8.419)
EL, $n^{-1/2}$ term for **D1**	10.711	10.974	10.709	12.617	6.637	21.189
	(6.386)	(5.826)	(4.899)	(8.352)	(7.569)	(9.293)
ULS, $n^{-1/2}$ term for **D2**	-1.586	-1.231	3.513	418.272	51.447	-191.318
	(2.346)	(2.325)	(2.506)	(456.232)	(172.005)	(187.744)
MLE, $n^{-1/2}$ term for **D2**	-3.556	-1.131	1.847	515.119	-15.938	-116.018
	(2.223)	(2.221)	(2.340)	(602.804)	(158.575)	(198.692)
AEL, $n^{-1/2}$ term for **D2**	10.943	12.916	14.168	-344.173	-360.641	-457.965
	(2.642)	(2.582)	(2.654)	(139.864)	(38.480)	(114.209)
EL, $n^{-1/2}$ term for **D2**	2.957	4.699	6.653	-346.174	-377.152	-527.335
	(2.436)	(2.509)	(2.526)	(152.169)	(40.698)	(104.709)
ULS, $n^{-1/2}$ term for **D3**	-2.855	5.382	-2.224	43.439	-4.101	-11.453
	(4.908)	(6.162)	(5.267)	(28.870)	(12.813)	(27.199)
MLE, $n^{-1/2}$ term for **D3**	-4.045	3.593	-2.980	33.784	-6.173	-15.590
	(4.666)	(6.086)	(5.137)	(25.665)	(11.716)	(26.387)
AEL, $n^{-1/2}$ term for **D3**	14.571	16.004	7.477	15.891	4.643	2.982
	(5.335)	(6.017)	(5.469)	(8.978)	(9.778)	(13.927)
EL, $n^{-1/2}$ term for **D3**	5.030	10.139	0.491	6.862	-4.348	-9.676
	(4.887)	(6.055)	(5.385)	(8.451)	(9.727)	(14.424)
ULS, $n^{-1/2}$ term for **D4**	-1.537	-3.597	-78.898	-83.325	-73.443	-83.267
	(3.442)	(3.814)	(6.305)	(6.606)	(7.674)	(6.757)
MLE, $n^{-1/2}$ term for **D4**	-0.646	-1.916	-73.879	-81.047	-73.431	-79.999
	(3.263)	(3.690)	(5.949)	(6.202)	(7.029)	(6.492)
AEL, $n^{-1/2}$ term for **D4**	15.953	12.382	-62.332	-68.599	-64.618	-68.598
	(4.087)	(4.058)	(5.717)	(5.677)	(6.720)	(5.920)
EL, $n^{-1/2}$ term for **D4**	8.044	6.073	-72.025	-76.064	-71.559	-76.978
	(3.832)	(3.950)	(6.180)	(6.314)	(7.538)	(6.570)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

Table 7: Confidence interval coverage.

The entries are regression estimates and their standard errors of the two-term expansion (36). The standard errors are corrected for clustering due to the same Monte Carlo sample.

Left limit, nominal 95%	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, leading term for **D1**	-1.155	-1.092	-1.224	0.384	1.261	0.044
	(2.632)	(2.439)	(2.648)	(2.486)	(2.569)	(2.535)
MLE, leading term for **D1**	-1.252	-1.526	-1.320	1.340	1.582	0.419
	(1.688)	(1.333)	(1.722)	(1.447)	(1.495)	(1.488)
AEL, leading term for **D1**	-0.092	-0.767	1.706	1.003	2.612	0.454
	(1.621)	(1.437)	(1.787)	(1.479)	(1.559)	(1.439)
EL, leading term for **D1**	0.210	-0.767	2.880	1.595	2.914	0.954
	(1.646)	(1.437)	(1.868)	(1.531)	(1.584)	(1.586)
ULS, leading term for **D2**	7.558**	8.127**	8.492**	7.245**	7.464**	8.810**
	(0.966)	(0.939)	(0.955)	(0.977)	(0.958)	(0.957)
MLE, leading term for **D2**	0.440	1.421	1.020	0.325	0.009	1.242
	(0.695)	(0.625)	(0.683)	(0.691)	(0.688)	(0.657)
AEL, leading term for **D2**	0.800	1.746	2.226*	0.431	0.703	1.541
	(0.774)	(0.709)	(0.746)	(0.793)	(0.755)	(0.621)
EL, leading term for **D2**	1.229	2.199*	2.809**	0.601	1.024	2.323*
	(0.797)	(0.725)	(0.770)	(0.808)	(0.781)	(0.737)
ELprofile, leading term for **D2**	0.653	-0.079	-1.451	-1.609	-1.139	-5.543*
	(2.011)	(1.926)	(2.159)	(2.220)	(2.184)	(2.145)
ULS, leading term for **D3**	0.298	-0.137	-0.076	-1.337	1.327	-0.853
	(2.178)	(2.170)	(2.134)	(2.060)	(2.034)	(2.061)
MLE, leading term for **D3**	1.850	0.461	1.129	0.219	2.247	-0.248
	(1.596)	(1.580)	(1.535)	(1.343)	(1.278)	(1.396)
AEL, leading term for **D3**	1.212	-0.018	-0.341	0.798	3.908*	-0.064
	(1.703)	(1.687)	(1.627)	(1.441)	(1.514)	(1.336)
EL, leading term for **D3**	2.009	0.670	-0.453	0.835	4.098*	-0.890
	(1.757)	(1.746)	(1.631)	(1.479)	(1.573)	(1.486)
ULS, leading term for **D4**	8.289**	12.394**	12.260**	12.372**	11.869**	9.282**
	(1.556)	(1.436)	(1.442)	(1.417)	(1.446)	(1.600)
MLE, leading term for **D4**	-1.012	2.759*	2.362*	3.326**	1.975	0.100
	(1.010)	(0.845)	(0.828)	(0.735)	(0.839)	(1.075)
AEL, leading term for **D4**	-0.481	3.102*	2.414*	3.385**	2.909**	0.765
	(1.086)	(0.954)	(0.847)	(0.856)	(0.827)	(0.928)
EL, leading term for **D4**	0.280	3.175**	2.785*	3.747**	3.191**	0.591
	(1.141)	(0.962)	(0.911)	(0.879)	(0.867)	(1.106)
ELprofile, leading term for **D4**	-1.524	3.567	4.939*	4.104	1.896	-10.683**
	(1.871)	(1.654)	(1.703)	(1.615)	(1.581)	(2.067)

Left limit, nominal 95%	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, $n^{-1/2}$ term for **D1**	-54.064	-39.677	-55.740	-73.860	-78.511	-68.242
	(42.450)	(39.096)	(42.726)	(40.648)	(42.228)	(41.297)
MLE, $n^{-1/2}$ term for **D1**	41.490	57.229**	39.814	4.309	15.744	22.324
	(24.215)	(16.176)	(24.874)	(21.978)	(22.628)	(21.734)
AEL, $n^{-1/2}$ term for **D1**	20.940	39.097	-19.956	5.674	-9.672	28.762
	(24.020)	(19.563)	(28.647)	(22.347)	(24.907)	(20.620)
EL, $n^{-1/2}$ term for **D1**	13.150	39.097	-46.339	-8.314	-17.462	5.081
	(24.804)	(19.563)	(30.925)	(23.946)	(25.646)	(24.259)
ULS, $n^{-1/2}$ term for **D2**	-159.489**	-171.549**	-173.557**	-157.195**	-157.392**	-181.828**
	(16.996)	(16.766)	(17.038)	(17.111)	(16.866)	(17.142)
MLE, $n^{-1/2}$ term for **D2**	15.549	2.943	9.482	18.249	22.552	4.459
	(10.190)	(9.325)	(10.182)	(10.052)	(9.880)	(9.835)
AEL, $n^{-1/2}$ term for **D2**	-3.791	-13.006	-21.937	-3.668	-3.044	6.614
	(11.876)	(11.049)	(11.852)	(12.150)	(11.533)	(9.222)
EL, $n^{-1/2}$ term for **D2**	-16.663	-22.624	-36.189*	-10.932	-13.321	-28.010
	(12.483)	(11.511)	(12.489)	(12.501)	(12.146)	(11.793)
ELprofile, $n^{-1/2}$ term for **D2**	-164.138**	-152.140**	-173.342**	-201.052**	-200.312**	-120.306**
	(33.217)	(31.776)	(35.271)	(36.303)	(35.809)	(34.320)
ULS, $n^{-1/2}$ term for **D3**	-190.794**	-184.980**	-177.624**	-159.668**	-203.830**	-165.094**
	(36.703)	(36.453)	(35.809)	(34.229)	(35.098)	(34.423)
MLE, $n^{-1/2}$ term for **D3**	-5.133	16.070	13.633	24.731	-9.159	33.457
	(24.688)	(23.416)	(22.943)	(18.790)	(19.588)	(19.313)
AEL, $n^{-1/2}$ term for **D3**	-4.378	10.705	26.626	5.871	-49.054	36.451
	(26.243)	(25.252)	(23.719)	(21.352)	(25.038)	(18.159)
EL, $n^{-1/2}$ term for **D3**	-21.985	-7.541	27.237	2.898	-56.868	36.677
	(27.739)	(26.862)	(23.723)	(22.120)	(26.174)	(20.637)
ULS, $n^{-1/2}$ term for **D4**	-205.728**	-256.946**	-251.166**	-251.795**	-250.123**	-229.190**
	(27.474)	(26.795)	(26.815)	(26.514)	(26.802)	(28.363)
MLE, $n^{-1/2}$ term for **D4**	43.072*	-6.251	7.441	-6.002	7.207	14.675
	(13.955)	(13.196)	(12.462)	(11.524)	(12.519)	(16.041)
AEL, $n^{-1/2}$ term for **D4**	20.018	-19.053	0.597	-20.113	-10.506	18.096
	(15.981)	(15.392)	(12.963)	(13.882)	(13.034)	(13.481)
EL, $n^{-1/2}$ term for **D4**	0.372	-22.044	-12.650	-28.439	-19.908	1.236
	(17.505)	(15.597)	(14.456)	(14.535)	(14.000)	(16.950)
ELprofile, $n^{-1/2}$ term for **D4**	-136.577**	-176.450**	-202.557**	-167.294**	-143.031**	-25.723
	(30.493)	(28.085)	(29.130)	(27.516)	(26.522)	(32.091)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

Left limit, nominal 95%	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, leading term for **D1**	2.423	1.443	2.250	-0.248	-0.534	1.430
	(2.507)	(2.413)	(2.438)	(2.549)	(2.466)	(2.447)
MLE, leading term for **D1**	2.340	2.718	2.940	0.375	-0.395	2.502
	(1.463)	(1.175)	(1.312)	(1.502)	(1.351)	(1.258)
AEL, leading term for **D1**	1.493	3.194*	2.968**	0.119	0.818	2.920
	(1.254)	(0.980)	(0.643)	(1.425)	(1.292)	(1.158)
EL, leading term for **D1**	2.292	4.062**	5.698**	-0.132	0.728	3.107
	(1.530)	(1.230)	(1.328)	(1.573)	(1.628)	(1.298)
ULS, leading term for **D2**	6.850**	8.296**	7.865**	11.829**	11.829**	11.820**
	(0.958)	(0.929)	(0.963)	(0.752)	(0.752)	(0.759)
MLE, leading term for **D2**	-0.163	1.096	0.128	4.563**	4.390**	4.493**
	(0.648)	(0.641)	(0.672)	(0.187)	(0.216)	(0.257)
AEL, leading term for **D2**	0.435	1.224	1.201	3.812**	3.996**	4.012**
	(0.572)	(0.567)	(0.545)	(0.377)	(0.377)	(0.377)
EL, leading term for **D2**	0.275	0.949	1.187	3.640**	3.993**	3.703**
	(0.712)	(0.693)	(0.677)	(0.416)	(0.403)	(0.409)
ELprofile, leading term for **D2**	0.175	1.751	0.600	1.646	0.733	-0.648
	(1.301)	(1.272)	(1.326)	(1.284)	(1.122)	(1.273)
ULS, leading term for **D3**	-0.111	-0.732	-0.543	3.477	0.381	0.188
	(2.004)	(2.085)	(2.123)	(1.680)	(1.993)	(1.950)
MLE, leading term for **D3**	1.296	-0.019	0.401	4.418**	1.590	1.658
	(1.242)	(1.414)	(1.473)	(0.587)	(1.304)	(1.255)
AEL, leading term for **D3**	1.683	0.617	0.251	4.608**	2.155	1.840
	(1.065)	(1.309)	(1.297)	(0.416)	(1.097)	(1.208)
EL, leading term for **D3**	1.275	0.023	1.276	4.881**	2.343	1.434
	(1.355)	(1.578)	(1.497)	(0.799)	(1.336)	(1.274)
ULS, leading term for **D4**	11.201**	11.316**	-31.664**	-30.615**	-31.126**	-29.527**
	(1.472)	(1.454)	(2.452)	(2.442)	(2.497)	(2.446)
MLE, leading term for **D4**	1.532	2.260*	-39.857**	-39.325**	-39.215**	-38.619**
	(0.867)	(0.850)	(2.188)	(2.190)	(2.262)	(2.208)
AEL, leading term for **D4**	2.743**	2.756**	-34.639**	-31.471**	-33.517**	-33.897**
	(0.671)	(0.738)	(2.020)	(2.005)	(2.110)	(2.049)
EL, leading term for **D4**	1.988	2.454*	-38.458**	-37.311**	-37.281**	-36.032**
	(0.859)	(0.901)	(2.203)	(2.178)	(2.231)	(2.205)
ELprofile, leading term for **D4**	1.562	1.891	-45.781**	-41.664**	-41.472**	-42.387**
	(1.082)	(1.088)	(2.351)	(2.343)	(2.381)	(2.382)

Left limit, nominal 95%	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, $n^{-1/2}$ term for **D1**	-110.327*	-73.107	-97.639	-73.238	-48.888	-82.710
	(41.869)	(39.717)	(40.608)	(41.518)	(39.796)	(40.366)
MLE, $n^{-1/2}$ term for **D1**	-16.390	2.606	-16.475	13.240	44.172	1.969
	(23.336)	(17.913)	(21.067)	(22.246)	(17.832)	(19.286)
AEL, $n^{-1/2}$ term for **D1**	17.217	4.561	9.706	25.877	31.166	0.675
	(18.065)	(14.635)	(7.258)	(20.261)	(17.878)	(17.812)
EL, $n^{-1/2}$ term for **D1**	-21.217	-20.977	-61.998*	11.785	6.009	-13.630
	(24.572)	(20.443)	(23.966)	(23.364)	(24.895)	(20.823)
ULS, $n^{-1/2}$ term for **D2**	-145.716**	-165.322**	-159.156**	-179.593**	-179.593**	-181.124**
	(16.726)	(16.607)	(16.986)	(14.791)	(14.791)	(14.895)
MLE, $n^{-1/2}$ term for **D2**	31.252**	11.327	30.167*	3.743	5.721	2.075
	(8.963)	(9.413)	(9.500)	(2.498)	(2.823)	(3.823)
AEL, $n^{-1/2}$ term for **D2**	39.206**	24.111*	29.821**	0.246	-0.783	0.082
	(7.483)	(7.861)	(7.333)	(5.679)	(5.766)	(5.754)
EL, $n^{-1/2}$ term for **D2**	19.635	5.302	6.750	-2.006	-2.701	0.062
	(10.425)	(10.392)	(10.132)	(6.349)	(6.266)	(6.199)
ELprofile, $n^{-1/2}$ term for **D2**	-11.790	-39.757	-22.276	-25.345	-3.315	8.759
	(20.242)	(20.573)	(20.938)	(20.528)	(17.139)	(19.197)
ULS, $n^{-1/2}$ term for **D3**	-165.151**	-169.420**	-177.228**	-203.981**	-173.100**	-164.036**
	(33.717)	(34.871)	(35.581)	(30.696)	(33.786)	(32.979)
MLE, $n^{-1/2}$ term for **D3**	22.839	29.763	18.232	-7.271	16.907	21.769
	(17.508)	(19.861)	(21.434)	(9.327)	(18.980)	(17.966)
AEL, $n^{-1/2}$ term for **D3**	22.548	29.038	31.364	-6.777	20.884	24.781
	(14.377)	(18.219)	(17.732)	(6.746)	(15.357)	(17.112)
EL, $n^{-1/2}$ term for **D3**	11.271	15.993	1.736	-24.276	4.767	23.005
	(19.905)	(23.223)	(22.633)	(13.852)	(20.239)	(18.165)
ULS, $n^{-1/2}$ term for **D4**	-235.580**	-242.845**	204.209**	193.581**	173.279**	176.921**
	(26.947)	(26.774)	(36.628)	(36.517)	(37.652)	(36.722)
MLE, $n^{-1/2}$ term for **D4**	17.469	-0.459	431.852**	426.100**	402.198**	410.946**
	(12.622)	(12.977)	(29.043)	(29.172)	(31.089)	(29.770)
AEL, $n^{-1/2}$ term for **D4**	11.938	5.756	425.105**	380.290**	379.210**	398.598**
	(9.620)	(11.040)	(25.370)	(25.740)	(28.069)	(26.369)
EL, $n^{-1/2}$ term for **D4**	6.998	-9.020	415.462**	402.620**	383.005**	369.691**
	(12.887)	(14.121)	(29.662)	(29.196)	(30.620)	(30.123)
ELprofile, $n^{-1/2}$ term for **D4**	-10.618	-24.998	439.117**	381.701**	359.986**	375.613**
	(16.986)	(17.405)	(32.743)	(32.968)	(33.971)	(33.888)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

Right limit, nominal 95%	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, leading term for **D1**	1.241	-1.352	-2.322	-0.154	-1.911	1.458
	(1.571)	(1.787)	(1.878)	(1.892)	(1.760)	(2.031)
MLE, leading term for **D1**	1.177	-1.422	-3.461	0.671	-0.750	1.315
	(1.702)	(1.840)	(1.987)	(1.995)	(1.859)	(2.165)
AEL, leading term for **D1**	1.703	-1.854	-0.702	2.200	-1.462	2.086
	(1.749)	(1.921)	(2.049)	(2.082)	(2.000)	(2.383)
EL, leading term for **D1**	1.602	-1.497	-0.691	2.491	-0.870	2.555
	(1.753)	(1.972)	(2.050)	(2.098)	(2.036)	(2.245)
ULS, leading term for **D2**	-1.074	-0.285	-0.289	-0.457	0.868	-0.766
	(0.858)	(0.878)	(0.910)	(0.857)	(0.868)	(0.900)
MLE, leading term for **D2**	-0.123	0.425	0.724	-0.525	1.344	0.303
	(0.881)	(0.906)	(0.940)	(0.893)	(0.897)	(0.928)
AEL, leading term for **D2**	0.877	0.589	-1.036	-0.355	1.497	2.080
	(0.915)	(0.955)	(1.060)	(1.002)	(0.991)	(1.042)
EL, leading term for **D2**	1.069	1.113	-0.568	-0.367	1.688	1.560
	(0.937)	(0.978)	(1.079)	(1.015)	(1.008)	(0.970)
ELprofile, leading term for **D2**	-2.241	-0.554	-0.383	-2.504	-3.925	-5.861
	(1.927)	(1.972)	(2.282)	(2.233)	(2.157)	(2.276)
ULS, leading term for **D3**	1.405	-0.689	2.716	-1.290	0.821	-2.621
	(1.566)	(1.797)	(1.810)	(1.815)	(1.746)	(2.075)
MLE, leading term for **D3**	0.976	-0.510	2.291	-0.773	1.338	-2.815
	(1.573)	(1.804)	(1.821)	(1.891)	(1.829)	(2.078)
AEL, leading term for **D3**	1.544	0.430	3.507	1.219	2.191	-0.578
	(1.744)	(1.938)	(1.978)	(1.931)	(1.891)	(2.321)
EL, leading term for **D3**	2.336	0.329	3.808	1.441	2.455	0.511
	(1.794)	(1.942)	(2.016)	(2.012)	(1.906)	(2.174)
ULS, leading term for **D4**	0.681	-8.084**	-7.724**	-6.568**	-3.964*	2.257
	(1.263)	(1.543)	(1.576)	(1.497)	(1.519)	(1.159)
MLE, leading term for **D4**	1.897	-7.544**	-6.405**	-6.458**	-2.498	3.439*
	(1.290)	(1.598)	(1.596)	(1.511)	(1.546)	(1.235)
AEL, leading term for **D4**	2.313	-7.828**	-5.382*	-5.216**	-2.865	5.493**
	(1.306)	(1.604)	(1.672)	(1.518)	(1.532)	(1.374)
EL, leading term for **D4**	2.760	-6.963**	-4.695*	-5.192**	-2.841	5.223**
	(1.336)	(1.658)	(1.701)	(1.546)	(1.576)	(1.290)
ELprofile, leading term for **D4**	-2.325	-12.326**	-10.877**	-12.368**	-8.219**	-2.244
	(1.856)	(2.074)	(2.082)	(2.062)	(2.036)	(1.991)

Right limit, nominal 95%	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, $n^{-1/2}$ term for **D1**	-26.301	10.881	24.096	-29.928	20.112	-38.296
	(24.954)	(26.798)	(27.790)	(29.982)	(25.844)	(32.988)
MLE, $n^{-1/2}$ term for **D1**	-25.241	9.238	34.486	-53.349	-4.660	-48.560
	(27.126)	(27.769)	(29.186)	(32.425)	(28.668)	(35.315)
AEL, $n^{-1/2}$ term for **D1**	-45.148	12.157	-33.492	-79.996	-15.341	-90.730
	(28.561)	(29.015)	(32.556)	(34.738)	(31.127)	(39.612)
EL, $n^{-1/2}$ term for **D1**	-46.292	0.664	-35.084	-86.194	-29.329	-80.244
	(28.615)	(30.294)	(32.594)	(35.169)	(32.200)	(37.404)
ULS, $n^{-1/2}$ term for **D2**	-1.067	-13.066	-19.462	-11.952	-37.158*	-15.063
	(13.003)	(13.581)	(14.180)	(13.182)	(13.801)	(13.908)
MLE, $n^{-1/2}$ term for **D2**	-22.160	-30.793	-43.796*	-17.859	-52.762**	-37.800
	(13.732)	(14.309)	(15.013)	(13.840)	(14.460)	(14.703)
AEL, $n^{-1/2}$ term for **D2**	-49.653**	-48.622*	-52.211*	-51.775*	-84.439**	-103.476**
	(14.663)	(15.275)	(16.816)	(15.949)	(16.213)	(17.196)
EL, $n^{-1/2}$ term for **D2**	-59.237**	-63.596**	-66.560**	-55.518**	-93.164**	-72.839**
	(15.125)	(15.823)	(17.264)	(16.176)	(16.565)	(15.803)
ELprofile, $n^{-1/2}$ term for **D2**	-108.217**	-143.637**	-250.772**	-196.952**	-156.714**	-134.828**
	(31.183)	(32.350)	(37.521)	(36.390)	(34.903)	(36.434)
ULS, $n^{-1/2}$ term for **D3**	-12.239	6.054	-46.481	13.370	-11.161	25.316
	(24.172)	(27.032)	(29.244)	(27.016)	(26.978)	(30.972)
MLE, $n^{-1/2}$ term for **D3**	-16.047	-7.548	-51.535	-11.636	-36.163	16.539
	(24.238)	(27.478)	(29.394)	(28.942)	(28.976)	(31.117)
AEL, $n^{-1/2}$ term for **D3**	-39.478	-45.319	-89.000*	-59.247	-63.012	-49.304
	(27.727)	(30.650)	(32.740)	(30.981)	(30.663)	(36.661)
EL, $n^{-1/2}$ term for **D3**	-55.835	-48.458	-97.426*	-68.767	-68.464	-45.440
	(29.054)	(30.734)	(33.524)	(32.460)	(31.057)	(34.470)
ULS, $n^{-1/2}$ term for **D4**	-23.547	77.241**	66.559*	48.797	16.813	-43.425
	(20.023)	(22.127)	(22.940)	(21.850)	(23.090)	(18.917)
MLE, $n^{-1/2}$ term for **D4**	-47.670	58.222	40.730	42.797	-15.004	-71.176**
	(21.010)	(23.473)	(23.818)	(22.209)	(24.130)	(20.731)
AEL, $n^{-1/2}$ term for **D4**	-62.859*	53.324	4.836	18.241	-13.752	-139.532**
	(21.527)	(23.587)	(25.676)	(22.821)	(23.822)	(23.942)
EL, $n^{-1/2}$ term for **D4**	-74.385**	28.984	-11.813	12.469	-25.501	-118.378**
	(22.215)	(24.955)	(26.423)	(23.399)	(24.688)	(22.461)
ELprofile, $n^{-1/2}$ term for **D4**	-109.657**	-5.340	-28.536	-19.028	-73.965	-168.465**
	(30.001)	(31.943)	(32.351)	(31.805)	(32.082)	(32.428)
F-test, leading terms	0.000	0.007	0.000	0.005	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

Right limit, nominal 95%	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, leading term for **D1**	-2.886	-2.070	-1.662	2.401	-1.260	0.682
	(2.168)	(2.329)	(2.153)	(2.101)	(2.451)	(2.131)
MLE, leading term for **D1**	-1.186	-2.282	-1.246	3.390	-1.802	3.682
	(2.236)	(2.322)	(2.155)	(2.250)	(2.493)	(2.224)
AEL, leading term for **D1**	2.557	0.965	3.232	6.491	5.812	10.436**
	(2.614)	(2.815)	(2.477)	(2.610)	(2.959)	(2.724)
EL, leading term for **D1**	2.220	1.004	3.684	6.266	4.895	9.267**
	(2.468)	(2.614)	(2.255)	(2.477)	(2.830)	(2.506)
ULS, leading term for **D2**	-0.087	0.297	1.335	-12.161**	-9.814**	-12.479**
	(1.012)	(0.973)	(0.927)	(1.590)	(1.559)	(1.585)
MLE, leading term for **D2**	-0.036	1.566	2.173	-9.295**	-7.298**	-9.456**
	(1.054)	(1.000)	(0.968)	(1.612)	(1.578)	(1.603)
AEL, leading term for **D2**	4.265**	6.227**	5.405**	-10.675**	-8.031**	-7.415**
	(1.280)	(1.225)	(1.228)	(1.735)	(1.718)	(1.709)
EL, leading term for **D2**	2.963	4.014**	4.749**	-10.714**	-7.885**	-8.235**
	(1.191)	(1.128)	(1.134)	(1.731)	(1.713)	(1.710)
ELprofile, leading term for **D2**	4.283	7.135**	8.190**	-6.344	-1.863	-1.562
	(1.870)	(1.702)	(1.766)	(2.669)	(2.632)	(2.614)
ULS, leading term for **D3**	2.842	0.180	-0.668	0.303	-0.247	0.412
	(2.049)	(2.090)	(2.011)	(2.430)	(2.463)	(2.435)
MLE, leading term for **D3**	1.207	0.586	0.422	-0.871	0.273	-0.501
	(2.053)	(2.088)	(2.065)	(2.459)	(2.545)	(2.474)
AEL, leading term for **D3**	7.984*	5.285	3.401	1.964	8.754*	8.105*
	(2.649)	(2.643)	(2.550)	(2.950)	(2.982)	(3.029)
EL, leading term for **D3**	8.369**	3.977	3.461	2.641	7.622*	7.494
	(2.412)	(2.453)	(2.340)	(2.823)	(2.895)	(2.918)
ULS, leading term for **D4**	-2.049	-6.400**	7.232**	7.044**	6.443**	6.896**
	(1.558)	(1.553)	(0.666)	(0.630)	(0.643)	(0.537)
MLE, leading term for **D4**	-1.416	-4.438*	7.626**	7.281**	6.326**	6.894**
	(1.611)	(1.598)	(0.727)	(0.650)	(0.633)	(0.537)
AEL, leading term for **D4**	2.133	0.851	13.065**	11.781**	11.675**	12.180**
	(1.901)	(1.932)	(1.091)	(0.988)	(1.028)	(1.046)
EL, leading term for **D4**	1.906	0.723	10.927**	10.445**	9.741**	10.303**
	(1.760)	(1.800)	(0.968)	(0.890)	(0.903)	(0.889)
ELprofile, leading term for **D4**	3.784	2.538	12.703**	11.446**	10.591**	11.171**
	(1.817)	(1.841)	(1.045)	(0.943)	(0.963)	(0.937)

Right limit, nominal 95%	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, $n^{-1/2}$ term for **D1**	-7.265	-22.133	-33.083	-88.923	-50.925	-54.936
	(33.472)	(36.738)	(34.034)	(35.244)	(39.401)	(34.685)
MLE, $n^{-1/2}$ term for **D1**	-42.341	-22.847	-41.340	-123.472*	-47.598	-118.474*
	(35.722)	(36.570)	(34.318)	(38.312)	(39.932)	(37.916)
AEL, $n^{-1/2}$ term for **D1**	-174.233**	-161.983**	-176.964**	-241.972**	-272.831**	-330.926**
	(44.241)	(46.840)	(42.380)	(45.525)	(50.477)	(48.387)
EL, $n^{-1/2}$ term for **D1**	-139.221**	-114.938*	-145.154**	-207.402**	-218.434**	-263.957**
	(41.585)	(43.286)	(38.774)	(43.279)	(48.180)	(44.796)
ULS, $n^{-1/2}$ term for **D2**	-53.624**	-51.441**	-64.679**	-166.689**	-193.378**	-161.234**
	(16.156)	(15.546)	(15.045)	(24.937)	(24.633)	(24.846)
MLE, $n^{-1/2}$ term for **D2**	-65.399**	-76.859**	-90.113**	-242.233**	-256.210**	-230.362**
	(16.915)	(16.306)	(15.977)	(25.504)	(25.122)	(25.343)
AEL, $n^{-1/2}$ term for **D2**	-232.752**	-248.673**	-242.042**	-454.669**	-483.383**	-511.645**
	(21.479)	(20.915)	(20.864)	(26.697)	(26.521)	(26.375)
EL, $n^{-1/2}$ term for **D2**	-171.152**	-166.090**	-186.463**	-425.972**	-463.020**	-476.136**
	(19.865)	(19.017)	(19.261)	(26.780)	(26.577)	(26.512)
ELprofile, $n^{-1/2}$ term for **D2**	-193.374**	-204.916**	-248.303**	-461.745**	-514.815**	-539.492**
	(31.792)	(29.875)	(31.220)	(42.085)	(41.657)	(41.477)
ULS, $n^{-1/2}$ term for **D3**	-78.282	-35.333	-26.424	-84.945	-85.595	-111.700*
	(33.509)	(32.887)	(31.239)	(38.980)	(39.396)	(39.378)
MLE, $n^{-1/2}$ term for **D3**	-65.004	-51.408	-59.484	-96.955	-129.682*	-127.918*
	(32.989)	(33.203)	(32.914)	(39.348)	(41.229)	(39.998)
AEL, $n^{-1/2}$ term for **D3**	-275.489**	-221.239**	-180.139**	-277.185**	-403.373**	-423.857**
	(45.076)	(44.237)	(42.280)	(48.456)	(50.126)	(50.650)
EL, $n^{-1/2}$ term for **D3**	-230.518**	-162.060**	-145.405**	-246.449**	-348.222**	-367.576**
	(41.589)	(40.794)	(38.869)	(46.584)	(48.715)	(49.033)
ULS, $n^{-1/2}$ term for **D4**	-46.863	34.674	-64.706**	-63.416**	-52.959**	-50.827**
	(24.718)	(23.089)	(13.287)	(12.723)	(12.446)	(11.171)
MLE, $n^{-1/2}$ term for **D4**	-65.015	0.094	-74.745**	-67.900**	-51.037**	-50.187**
	(25.843)	(24.570)	(14.423)	(13.172)	(12.215)	(11.154)
AEL, $n^{-1/2}$ term for **D4**	-216.457**	-187.651**	-217.169**	-185.351**	-186.000**	-190.698**
	(31.734)	(31.913)	(22.261)	(20.414)	(20.887)	(21.256)
EL, $n^{-1/2}$ term for **D4**	-164.325**	-141.934**	-160.581**	-149.686**	-137.526**	-138.384**
	(29.353)	(29.667)	(19.687)	(18.468)	(18.301)	(18.270)
ELprofile, $n^{-1/2}$ term for **D4**	-210.539**	-182.904**	-199.509**	-172.277**	-159.446**	-156.395**
	(30.686)	(30.745)	(21.460)	(19.658)	(19.547)	(19.319)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

Two-sided, nominal 90%	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, leading term for **D1**	-1.971	-3.038	-3.871	-2.696	-1.806	-0.840
	(1.735)	(1.726)	(1.816)	(1.765)	(1.741)	(1.782)
MLE, leading term for **D1**	0.393	-1.245	-2.449	0.276	1.283	1.323
	(1.408)	(1.392)	(1.537)	(1.452)	(1.376)	(1.493)
AEL, leading term for **D1**	0.695	-1.253	-0.618	0.872	0.405	1.182
	(1.405)	(1.437)	(1.557)	(1.477)	(1.460)	(1.569)
EL, leading term for **D1**	0.590	-1.225	-0.232	1.190	0.666	1.693
	(1.415)	(1.455)	(1.572)	(1.489)	(1.474)	(1.541)
ULS, leading term for **D2**	1.990*	2.619**	2.897**	2.036*	2.826**	2.501**
	(0.724)	(0.718)	(0.725)	(0.724)	(0.718)	(0.730)
MLE, leading term for **D2**	0.060	0.960	0.849	-0.248	0.372	0.512
	(0.663)	(0.651)	(0.670)	(0.666)	(0.667)	(0.667)
AEL, leading term for **D2**	0.089	0.527	-0.866	-1.572	-0.477	0.770
	(0.694)	(0.690)	(0.741)	(0.739)	(0.723)	(0.703)
EL, leading term for **D2**	0.054	0.829	-0.625	-1.749	-0.500	0.903
	(0.708)	(0.701)	(0.752)	(0.749)	(0.735)	(0.701)
ELprofile, leading term for **D2**	-9.644**	-9.565**	-14.080**	-16.043**	-15.761**	-18.919**
	(1.506)	(1.499)	(1.619)	(1.630)	(1.615)	(1.645)
ULS, leading term for **D3**	-6.545**	-8.353**	-6.040**	-9.406**	-6.604**	-9.671**
	(1.645)	(1.701)	(1.659)	(1.693)	(1.634)	(1.749)
MLE, leading term for **D3**	2.190	0.233	2.506	-0.367	2.059	-1.445
	(1.310)	(1.393)	(1.346)	(1.386)	(1.308)	(1.466)
AEL, leading term for **D3**	1.518	-0.718	1.523	0.106	2.662	-0.727
	(1.404)	(1.483)	(1.451)	(1.431)	(1.383)	(1.542)
EL, leading term for **D3**	2.180	-0.838	1.498	0.085	2.756	-0.364
	(1.426)	(1.503)	(1.467)	(1.468)	(1.404)	(1.513)
ULS, leading term for **D4**	2.825	-0.324	-0.133	0.275	1.770	4.078**
	(1.097)	(1.170)	(1.177)	(1.144)	(1.145)	(1.075)
MLE, leading term for **D4**	0.959	-3.009*	-2.277	-2.084	-0.478	2.011
	(0.950)	(1.061)	(1.047)	(1.005)	(1.018)	(0.942)
AEL, leading term for **D4**	0.674	-3.493*	-2.488	-1.943	-0.559	2.709*
	(0.981)	(1.085)	(1.085)	(1.029)	(1.014)	(0.953)
EL, leading term for **D4**	1.041	-3.299*	-2.242	-1.937	-0.880	2.450
	(1.002)	(1.105)	(1.106)	(1.045)	(1.043)	(0.966)
ELprofile, leading term for **D4**	-10.707**	-13.771**	-12.154**	-13.651**	-12.903**	-18.032**
	(1.413)	(1.439)	(1.441)	(1.429)	(1.412)	(1.509)

Two-sided, nominal 90%	λ_2 b/se	λ_3 b/se	λ_4 b/se	λ_5 b/se	λ_6 b/se	ϕ_{11} b/se
ULS, n^{-1} term for **D1**	-0.668	-0.273	-0.378	-0.798	-0.566	-0.977*
	(0.357)	(0.342)	(0.365)	(0.363)	(0.357)	(0.378)
MLE, $1000 \cdot n^{-1}$ term for **D1**	0.123	0.552	0.515	-0.294	0.052	-0.280
	(0.261)	(0.227)	(0.269)	(0.286)	(0.260)	(0.303)
AEL, $1000 \cdot n^{-1}$ term for **D1**	-0.131	0.413	-0.384	-0.516	-0.183	-0.569
	(0.270)	(0.248)	(0.311)	(0.302)	(0.286)	(0.328)
EL, $1000 \cdot n^{-1}$ term for **D1**	-0.187	0.325	-0.597	-0.672	-0.346	-0.650
	(0.274)	(0.257)	(0.322)	(0.311)	(0.296)	(0.326)
ULS, $1000 \cdot n^{-1}$ term for **D2**	-1.241**	-1.414**	-1.514**	-1.304**	-1.490**	-1.514**
	(0.155)	(0.157)	(0.159)	(0.156)	(0.158)	(0.160)
MLE, $1000 \cdot n^{-1}$ term for **D2**	-0.035	-0.191	-0.281	0.017	-0.202	-0.235
	(0.124)	(0.125)	(0.131)	(0.123)	(0.128)	(0.129)
AEL, $1000 \cdot n^{-1}$ term for **D2**	-0.392*	-0.457**	-0.577**	-0.406*	-0.623**	-0.716**
	(0.137)	(0.137)	(0.148)	(0.145)	(0.145)	(0.144)
EL, $1000 \cdot n^{-1}$ term for **D2**	-0.559**	-0.650**	-0.796**	-0.485**	-0.770**	-0.743**
	(0.142)	(0.143)	(0.153)	(0.147)	(0.149)	(0.144)
ELprofile, $1000 \cdot n^{-1}$ term for **D2**	-1.996**	-2.135**	-3.183**	-2.844**	-2.597**	-1.886**
	(0.306)	(0.305)	(0.318)	(0.319)	(0.318)	(0.321)
ULS, $1000 \cdot n^{-1}$ term for **D3**	-0.996*	-0.818	-1.196**	-0.526	-1.050*	-0.566
	(0.322)	(0.326)	(0.331)	(0.314)	(0.320)	(0.328)
MLE, $1000 \cdot n^{-1}$ term for **D3**	-0.155	0.057	-0.329	0.143	-0.295	0.340
	(0.254)	(0.259)	(0.270)	(0.251)	(0.256)	(0.263)
AEL, $1000 \cdot n^{-1}$ term for **D3**	-0.340	-0.233	-0.510	-0.321	-0.806*	-0.163
	(0.279)	(0.286)	(0.295)	(0.278)	(0.291)	(0.300)
EL, $1000 \cdot n^{-1}$ term for **D3**	-0.610	-0.375	-0.572	-0.434	-0.912*	-0.128
	(0.294)	(0.294)	(0.300)	(0.290)	(0.298)	(0.293)
ULS, $1000 \cdot n^{-1}$ term for **D4**	-1.833**	-1.480**	-1.541**	-1.603**	-1.893**	-2.144**
	(0.245)	(0.250)	(0.252)	(0.247)	(0.253)	(0.247)
MLE, $1000 \cdot n^{-1}$ term for **D4**	-0.083	0.326	0.274	0.280	-0.122	-0.449
	(0.183)	(0.192)	(0.192)	(0.179)	(0.197)	(0.192)
AEL, $1000 \cdot n^{-1}$ term for **D4**	-0.340	0.201	-0.035	-0.000	-0.205	-0.902**
	(0.195)	(0.200)	(0.207)	(0.193)	(0.197)	(0.204)
EL, $1000 \cdot n^{-1}$ term for **D4**	-0.588*	-0.015	-0.271	-0.113	-0.360	-0.884**
	(0.206)	(0.210)	(0.217)	(0.199)	(0.206)	(0.206)
ELprofile, $1000 \cdot n^{-1}$ term for **D4**	-1.900**	-1.421**	-1.835**	-1.389**	-1.539**	-1.577**
	(0.284)	(0.284)	(0.287)	(0.282)	(0.280)	(0.294)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

Two-sided, noiminal 90%	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, leading term for **D1**	-3.590	-2.716	-3.059	-2.306	-3.989	-1.468
	(1.843)	(1.840)	(1.812)	(1.822)	(1.900)	(1.790)
MLE, leading term for **D1**	-0.743	-0.070	-0.372	0.380	-2.047	2.820
	(1.553)	(1.514)	(1.490)	(1.553)	(1.636)	(1.454)
AEL, leading term for **D1**	-0.719	-0.277	0.694	0.186	0.053	3.805
	(1.664)	(1.696)	(1.519)	(1.694)	(1.812)	(1.648)
EL, leading term for **D1**	-0.369	1.431	2.977	0.283	-0.043	4.329*
	(1.649)	(1.612)	(1.482)	(1.665)	(1.796)	(1.563)
ULS, leading term for **D2**	1.246	2.491**	2.884**	-10.516**	-9.050**	-10.799**
	(0.768)	(0.740)	(0.734)	(0.982)	(0.967)	(0.982)
MLE, leading term for **D2**	-1.158	0.782	0.524	-12.014**	-10.606**	-11.957**
	(0.723)	(0.688)	(0.687)	(0.971)	(0.955)	(0.968)
AEL, leading term for **D2**	-0.730	1.005	0.454	-20.600**	-18.598**	-19.001**
	(0.804)	(0.771)	(0.774)	(1.060)	(1.050)	(1.049)
EL, leading term for **D2**	-1.005	0.329	0.732	-20.034**	-17.886**	-19.122**
	(0.789)	(0.754)	(0.752)	(1.057)	(1.046)	(1.049)
ELprofile, leading term for **D2**	-1.481	1.700	0.833	-19.357**	-16.699**	-18.441**
	(1.276)	(1.180)	(1.227)	(1.657)	(1.634)	(1.643)
ULS, leading term for **D3**	-6.613**	-8.820**	-9.679**	-7.026**	-9.627**	-9.950**
	(1.699)	(1.752)	(1.766)	(1.746)	(1.843)	(1.839)
MLE, leading term for **D3**	1.232	-0.021	-0.547	0.196	-1.463	-2.331
	(1.402)	(1.461)	(1.488)	(1.503)	(1.645)	(1.624)
AEL, leading term for **D3**	2.371	0.479	-0.748	-2.318	-0.428	-1.965
	(1.631)	(1.671)	(1.647)	(1.771)	(1.818)	(1.876)
EL, leading term for **D3**	3.413	-0.112	0.511	-0.834	-0.098	-1.343
	(1.549)	(1.646)	(1.582)	(1.717)	(1.794)	(1.825)
ULS, leading term for **D4**	1.371	-0.771	-18.650**	-18.146**	-19.381**	-17.247**
	(1.162)	(1.179)	(1.478)	(1.467)	(1.495)	(1.461)
MLE, leading term for **D4**	-1.132	-2.032	-20.515**	-20.349**	-21.256**	-19.883**
	(1.059)	(1.057)	(1.389)	(1.381)	(1.407)	(1.376)
AEL, leading term for **D4**	-0.927	-1.293	-14.098**	-12.763**	-14.801**	-14.301**
	(1.159)	(1.181)	(1.353)	(1.318)	(1.373)	(1.355)
EL, leading term for **D4**	-0.541	-0.917	-18.582**	-18.241**	-19.021**	-17.679**
	(1.118)	(1.145)	(1.417)	(1.397)	(1.418)	(1.405)
ELprofile, leading term for **D4**	-0.785	-1.246	-24.518**	-22.874**	-23.754**	-23.340**
	(1.180)	(1.193)	(1.506)	(1.484)	(1.500)	(1.500)

Two-sided, noiminal 90%	θ_1 b/se	θ_2 b/se	θ_3 b/se	θ_4 b/se	θ_5 b/se	θ_6 b/se
ULS, n^{-1} term for **D1**	-0.949	-0.874	-1.015*	-1.274**	-0.915	-1.130*
	(0.381)	(0.383)	(0.377)	(0.385)	(0.393)	(0.379)
MLE, $1000 \cdot n^{-1}$ term for **D1**	-0.395	-0.171	-0.341	-0.782	-0.085	-0.878*
	(0.310)	(0.297)	(0.294)	(0.325)	(0.318)	(0.316)
AEL, $1000 \cdot n^{-1}$ term for **D1**	-1.126*	-1.211**	-1.099**	-1.584**	-1.915**	-2.483**
	(0.353)	(0.364)	(0.326)	(0.371)	(0.398)	(0.386)
EL, $1000 \cdot n^{-1}$ term for **D1**	-1.149*	-1.092*	-1.459**	-1.425**	-1.711**	-2.083**
	(0.352)	(0.350)	(0.336)	(0.363)	(0.393)	(0.368)
ULS, $1000 \cdot n^{-1}$ term for **D2**	-1.555**	-1.667**	-1.719**	-2.563**	-2.738**	-2.518**
	(0.166)	(0.163)	(0.163)	(0.196)	(0.195)	(0.196)
MLE, $1000 \cdot n^{-1}$ term for **D2**	-0.263	-0.496**	-0.441*	-1.703**	-1.777**	-1.625**
	(0.140)	(0.138)	(0.136)	(0.192)	(0.190)	(0.191)
AEL, $1000 \cdot n^{-1}$ term for **D2**	-1.493**	-1.699**	-1.591**	-3.277**	-3.509**	-3.659**
	(0.169)	(0.167)	(0.166)	(0.199)	(0.198)	(0.197)
EL, $1000 \cdot n^{-1}$ term for **D2**	-1.171**	-1.212**	-1.349**	-3.081**	-3.378**	-3.388**
	(0.164)	(0.159)	(0.160)	(0.200)	(0.198)	(0.198)
ELprofile, $1000 \cdot n^{-1}$ term for **D2**	-1.541**	-1.814**	-2.004**	-3.528**	-3.759**	-3.784**
	(0.272)	(0.263)	(0.271)	(0.315)	(0.314)	(0.314)
ULS, $1000 \cdot n^{-1}$ term for **D3**	-1.319**	-1.016*	-0.954*	-1.631**	-1.440**	-1.502**
	(0.341)	(0.341)	(0.341)	(0.356)	(0.367)	(0.367)
MLE, $1000 \cdot n^{-1}$ term for **D3**	-0.308	-0.173	-0.272	-0.714	-0.841	-0.710
	(0.276)	(0.282)	(0.289)	(0.305)	(0.335)	(0.324)
AEL, $1000 \cdot n^{-1}$ term for **D3**	-1.924**	-1.490**	-1.107*	-2.000**	-2.840**	-2.944**
	(0.359)	(0.356)	(0.342)	(0.374)	(0.393)	(0.400)
EL, $1000 \cdot n^{-1}$ term for **D3**	-1.689**	-1.134**	-1.074*	-1.934**	-2.577**	-2.545**
	(0.344)	(0.344)	(0.331)	(0.367)	(0.388)	(0.390)
ULS, $1000 \cdot n^{-1}$ term for **D4**	-2.209**	-1.639**	0.637	0.588	0.476	0.539
	(0.257)	(0.252)	(0.278)	(0.276)	(0.283)	(0.276)
MLE, $1000 \cdot n^{-1}$ term for **D4**	-0.386	-0.039	2.353**	2.374**	2.288**	2.376**
	(0.210)	(0.201)	(0.228)	(0.224)	(0.235)	(0.224)
AEL, $1000 \cdot n^{-1}$ term for **D4**	-1.560**	-1.449**	1.217**	1.160**	1.109**	1.231**
	(0.247)	(0.250)	(0.242)	(0.234)	(0.248)	(0.242)
EL, $1000 \cdot n^{-1}$ term for **D4**	-1.207**	-1.196**	1.543**	1.590**	1.509**	1.416**
	(0.236)	(0.241)	(0.252)	(0.244)	(0.251)	(0.250)
ELprofile, $1000 \cdot n^{-1}$ term for **D4**	-1.721**	-1.639**	1.415**	1.269**	1.193**	1.284**
	(0.253)	(0.254)	(0.275)	(0.270)	(0.275)	(0.275)
F-test, leading terms	0.000	0.000	0.000	0.000	0.000	0.000
F-test, $n^{-1/2}$ terms	0.000	0.000	0.000	0.000	0.000	0.000

Table 8: Simulation results for the bootstrap approximation of Bartlett correction.

Sample sizes	100	200	500	1000
Number of Monte Carlo samples	400	600	600	300
EL	0.275	0.117	0.051	0.065
	0.000	0.000	0.079	0.139
FIML	0.428	0.460	0.522	0.476
	0.000	0.000	0.000	0.000
Bartlett corrected, $B = 200$	0.062	0.044	0.044	0.047
	0.078	0.178	0.179	0.496
Bartlett corrected, $B = 50$	0.067	0.074	0.038	0.036
	0.043	0.126	0.322	0.805
Direct bootstrap	0.085	0.024	0.033	0.053
	0.004	0.860	0.496	0.331

Reported entries are Kolmogorov-Smirnov distances D and p-values of the test against the asymptotic χ^2_9 distribution, except the direct bootstrap estimate which is referred to the uniform distribution.