Specification testing with Heywood cases: Can standard errors be trusted?

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What is a Heywood case?

Heywood (1931) considered characterization of the correlation matrices:

$$\begin{pmatrix} 1 & r_1 r_2 & r_1 r_3 & \dots & r_1 r_n \\ r_1 r_2 & 1 & r_2 r_3 & \dots & r_2 r_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1 r_n & r_2 r_n & r_3 r_n & \dots & 1 \end{pmatrix}, r_1 \geq r_2 \geq \ldots \geq r_n \geq 0$$

He showed that for this scheme

$$r_1^2 \leq 1 + \frac{1}{\frac{r_2^2}{1-r_2^2} + \frac{r_3^2}{1-r_3^2} + \dots \frac{r_n^2}{1-r_n^2}}$$

and hence r_1 can be greater than 1, which was an extension on the earlier results by Spearman and Garnett.

Causes of improper solutions

- Outliers (Bollen 1987).
- Non-convergence and underidentification (Van Driel 1978, Boomsma & Hoogland 2001).
- Empirical underidentification (Rindskopf 1984).
- Structurally misspecified models (Van Driel 1978, Dillon, Kumar & Mulani 1987, Sato 1987, Bollen 1989, Kolenikov & Bollen 2008).
- Sampling fluctuations (Van Driel 1978, Boomsma 1983, Anderson & Gerbing 1984).

Hypotheses on variances

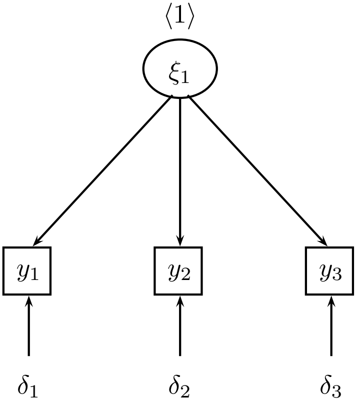
$H_+ : \sigma^2 > 0$	“Normal” situation
$H_0 : \sigma^2 = 0$	Perfect indicator (!?) or no factor present
$H_- : \sigma^2 < 0$	Misspecified model

	Sample estimate	
Population value	$\hat{\sigma}^2 > 0$	$\hat{\sigma}^2 < 0$
$\sigma^2 > 0$	☺	??
$\sigma^2 = 0$	☺	??
$\sigma^2 < 0$	Type I error	Misspecification detected!

Research question: testing specification, as expressed via

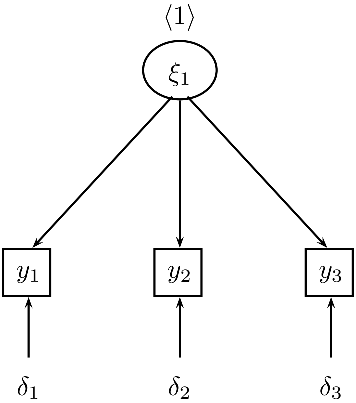
$$H_0 \cap H_+ \text{ against } H_-$$

Is that really a variance?



$$
\begin{aligned}
\sigma_{11} &= \lambda_1^2 + \theta_1 \\
\sigma_{12} &= \lambda_1 \lambda_2 \\
\sigma_{13} &= \lambda_1 \lambda_3 \\
\sigma_{22} &= \lambda_2^2 + \theta_2 \\
\sigma_{23} &= \lambda_2 \lambda_3 \\
\sigma_{33} &= \lambda_3^2 + \theta_3
\end{aligned}
$$

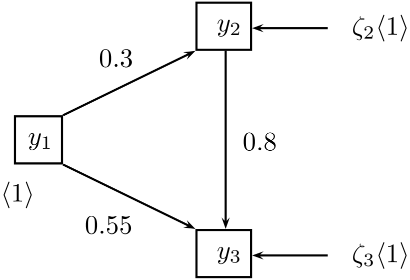
Is that really a variance?



$$\begin{aligned}
\hat\lambda_1 &= \sqrt{s_{12}s_{13}/s_{23}} \\
\hat\lambda_2 &= \sqrt{s_{12}s_{23}/s_{13}} \\
\hat\lambda_3 &= \sqrt{s_{13}s_{23}/s_{12}} \\
\hat\theta_1 &= s_{11} - s_{12}s_{13}/s_{23} \\
\hat\theta_2 &= s_{22} - s_{12}s_{23}/s_{13} \\
\hat\theta_3 &= s_{33} - s_{13}s_{23}/s_{12}
\end{aligned}$$

Population Heywood case

Data generating process



Jump to simulations

Population Heywood case

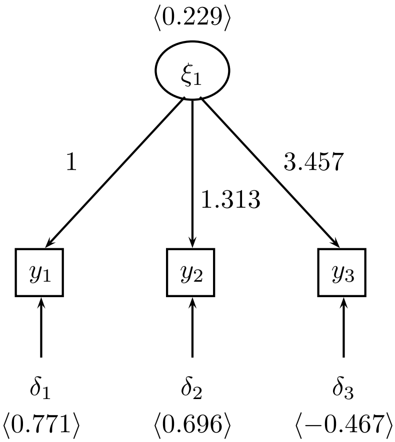
Covariance matrix

$$\begin{pmatrix} 1 & 0.3 & 0.79 \\ 0.3 & 1.09 & 1.037 \\ 0.79 & 1.037 & 2.264 \end{pmatrix}$$

Jump to simulations

Population Heywood case

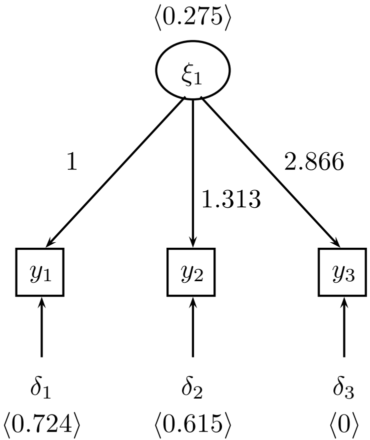
CFA model fit



Jump to simulations

Population Heywood case

CFA with restricted variance



Jump to simulations

Misspecified models

If the model is misspecified, doesn't everything just fall apart??

- Distributional misspecification: the distribution of the data is not normal, likelihood methods are not fully applicable (Satorra 1990, Satorra & Bentler 1994).
- Structural misspecification: the structure of the model, the number of latent variables, the relations between the variables in the model are not specified correctly (Yuan, Marshall & Bentler 2003, Yuan & Hayashi 2006)
- Heywood case: impossible value in the population; evidence of structural misspecification (or what??)

Misspecified models

Huber (1967) and White (1982):

- Point estimates are consistent for the minimizer of the population fit function:

$$\arg\min F(S, \Sigma(\theta)) \to \arg\min F(\Sigma, \Sigma(\theta)) = \theta_0,$$
$$F_{ML}(\Omega, \Sigma(\theta)) = \ln|\Sigma(\theta)| + \Sigma(\theta)^{-1}\Omega - \ln|\Omega| - \dim \Sigma(\theta)$$
$$F(S, \Sigma(\theta)) \stackrel{as.eq.}{=} (s - \sigma(\theta))' \, V_{NT} \, (s - \sigma(\theta)),$$
$$s = \mathrm{vech}\, S, \sigma(\theta) = \mathrm{vech}\, \Sigma(\theta)$$

- Uncertainty about these estimates is given by an asymptotic covariance matrix (sandwich estimator):

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \stackrel{d}{\longrightarrow} N\bigl(0, A^{-1} B A^{-T}\bigr),$$
$$A = \mathbb{E}\, \partial \psi(X, \theta_0), \quad B = \mathbb{E}\, \psi(X, \theta_0) \psi(X, \theta_0)^T,$$
$$\psi(X, \theta_0) = -\dot{\sigma}(\theta) V(s - \sigma(\theta))$$

Sandwich variance estimator

Empirical estimates of “bread” A and “meat” B:

$$\hat{A} = \frac{1}{n}\sum_{i=1}^{n} \partial\psi(x_i, \hat{\theta}),$$

$$\hat{B} = \frac{1}{n}\sum_{i=1}^{n} \psi(x_i, \hat{\theta})\psi(x_i, \hat{\theta})^T$$

- Easily doable with analytic or numeric derivatives
- Byproduct of optimization problem
- Makes fewer assumptions, robust to broader range of model assumption violations
- Studied in and out in regression and econometrics literature
- Easily extendable to clustered/complex survey data

Sandwich variance estimator

Special cases:

- Eicker (1967) and White (1980), linear regression with heteroskedastic errors:

$$\hat{\beta} = (X'X)^{-1}X'y, \quad \hat{v}(\hat{\beta}) = (X'X)^{-1}(X'ee'X)(X'X)^{-1}$$

- Browne (1974): least squares estimates for SEM
- White (1982): somewhat milder regularity conditions that are easier to check in practice; application to some common econometric models.
- Arminger & Schoenberg (1989): an econometrics paper in *Psychometrika*
- Satorra & Bentler (1994): model based sandwich estimator with explicit expression for matrix B (the meat of the sandwich) as a function of the model parameters and the fourth order moments of data
- Yuan & Hayashi (2006): comparison of empirical sandwich and the bootstrap standard errors for SEM

Tests of $H_0 \cap H_+$ vs. H_-

Likelihood ratio type tests

- Overall fit
- χ^2-difference tests (from $\theta_k = 0$)
- Signed root tests for simple χ^2 difference ΔT and scaled Satorra & Bentler (2001) difference ΔT_{sc}

$$r(\theta_0) = \mathrm{sign}[\hat{\theta} - \theta_0]\sqrt{\Delta T_{sc}}$$

Wald type tests, using

- Information matrix standard errors
- Satorra-Bentler standard errors
- Huber empirical sandwich standard errors
- Bootstrap standard errors (empirical, Bollen-Stine)

Can standard errors be trusted?

Distributional specification	Structural specification: Correct	Structural specification: Incorrect
Correct	IM, S-B, ES, EB, BSB	IM, ES, EB
Incorrect	S-B, ES, EB, BSB	ES, EB

Analytic standard errors:

- **IM** observed or expected information matrix
- **S-B** Satorra-Bentler standard errors
- **ES** empirical (Huber) sandwich

Resampling standard errors:

- **EB** empirical bootstrap
- **BSB** Bollen-Stine rotating bootstrap

Simulation study

Saturated model with 3 variables and 6 parameters: $\chi^2_0 \equiv 0$, no way to test the model fit... unless you hit a Heywood case!

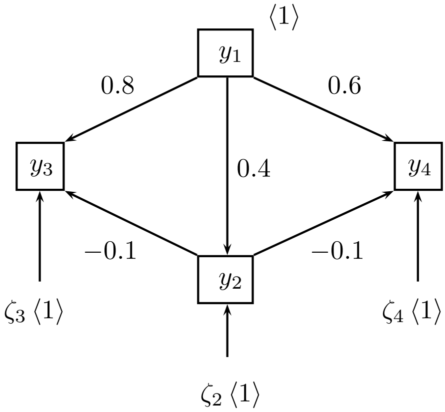
Jump to the Heywood case in population example

Main results:

- HUGE biases of the Heywood case in small samples.
- Non-normal data: information matrix standard errors are biased leading to undercoverage of CIs

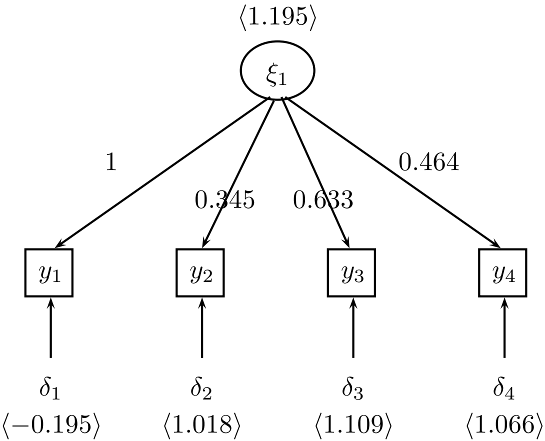
Simulation study

Larger model with 4 variables, 8 parameters and 2 d.f.
Data generating process



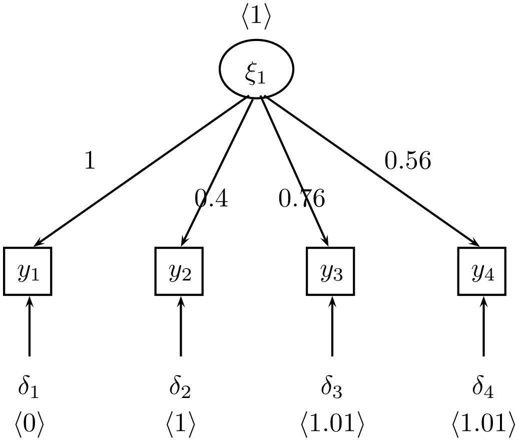
Simulation study

Larger model with 4 variables, 8 parameters and 2 d.f. CFA model fit



Simulation study

Larger model with 4 variables, 8 parameters and 2 d.f. CFA with restricted variance



Simulation study

Larger model with 4 variables, 8 parameters and 2 d.f.
Main results:

- HUGE biases of the Heywood case in small samples.
- Information matrix standard errors are biased leading to undercoverage of CIs based on them when the data are non-normal.
- Satorra-Bentler standard errors are biased leading to undercoverage *even when the data are normal!*
- Huber empirical sandwich standard errors are asymptotically accurate.
- Empirical bootstrap standard errors are even more accurate in smaller samples, and better centered to the true variability.
- Signed root of Satorra-Bentler scaled difference is the most accurate test in terms of the size, and has the greatest power among accurate tests!

Conclusions and recommendations

- Heywood cases are possible in population if the model is grossly misspecified.
- Testing Heywood cases might be the only way to test specification in exactly identified models.
- Empirical sandwich and empirical bootstrap standard errors are the only ones asymptotically accurate under double misspecification.
- Signed root of scaled difference test is a viable alternative to unwieldy likelihood ratio tests when testing at the boundary

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